

### Laboratoire d'Arithmétique, de Calcul formel et d'Optimisation ESA - CNRS 6090

# SKETCHES AND SPECIFICATIONS USER'S GUIDE

## First part: Wefts for Explicit Specifications

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Rapport de recherche n° 2000-03

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### First part: Wefts for Explicit Specification

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February 17, 2000

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#### SKETCHES AND SPECIFICATIONS — USER'S GUIDE

SKETCHES AND SPECIFICATIONS is a common denomination for several papers which deal with applications of Ehresmann's sketch theory to computer science. These papers can be considered as the first steps towards a unified theory for software engineering. However, their aim is not to advocate a unification of computer languages; they are designed to build a frame for the study of notions which arise from several areas in computer science.

These papers are arranged in two complementary families:

#### REFERENCE MANUAL and USER'S GUIDE.

The reference manual provides general definitions and results, with comprehensive proofs. On the other hand, the user's guide places emphasis on motivations and gives a detailed description of several examples. These two families, though complementary, can be read independently. No prerequisite is assumed; however, it can prove helpful to be familiar either with specification techniques in computer science or with category theory in mathematics.

These papers are under development, they are, or will be, available at: http://www.unilim.fr/laco/rapports.

#### REFERENCE MANUAL:

First Part: Compositive Graphs Second Part: Projective Sketches

Third Part: Models

#### USER'S GUIDE:

First Part: Wefts for Explicit Specification Second Part: Mosaics for Implicit Specification Third Part: Functional and Imperative Programs

In addition, further papers about APPLICATIONS are in progress, with several co-authors. They deal with various topics, including the notion of state in computer science [state], overloading, coercions and subsorts.

These articles owe a great deal to the working group *sketches and computer algebra*; we would like to thank its participants, specially Catherine Oriat and Jean-Claude Reynaud, as well as the CNRS.

These papers have been processed with LATEX and Xy-pic.

#### First Part: Wefts for Explicit Specification

The aim of this paper is to define wefts (for the French "trames") and to show how they can be used for specification issues in computer science. Moreover, in [guide2] and [guide3], wefts will be extended to mosaics, in order to deal with implicit features of computer languages, including the notion of state. Here and in [guide2], we focus on the realizations of the wefts and mosaics, which give them their meaning. Programs are considered in [guide3]: wefts provide a good frame for functional programming, while mosaics extends it to imperative programming.

This paper is part of a general study of some applications of Ehresmann's sketch theory to computer science, along with the reference manual [ref]. No prerequisite is assumed to read it.

#### 1 Introduction

Specifications are used in industry in order to design software. The client's requirements are used in order to build a specification, which in turn is used for making and testing the implementation. Hence, the aim of a specification is twofold: it must help to clarify the requirements and to check that the software does meet them.

Among the range of specifications are algebraic specifications [Goguen et al. 78, Wirsing 90, Astesiano et al. 99]. With each algebraic specification are associated on one side its models and on the other side its terms. Models are used to check that the problem is well-posed and terms are used to describe programs and evaluation processes, at least in functional (or applicative) languages like Lisp and ML.

The use of algebraic specifications for *imperative* languages like Pascal or C is less straightforward, because of the *implicit* features of these languages. These implicit features include the notion of state and the side-effects, as well as overloading and coercions, or error-handling.

In this paper, we introduce a new way to describe and handle specifications, which can be considered as an extension of the theory of algebraic specifications. Then, using this new notion of specification, the problems involved with implicit features will be dealt with in [guide2] and [guide3].

It is known [Lellahi 89, Wells & Barr 88] that Ehresmann's *sketches* theory [Ehresmann 66, Ehresmann 67a, Ehresmann 67b, Ehresmann 68] gives rise to notions which are quite similar to algebraic specifications. *Wefts* and *patchworks* theory, as introduced by Lair in [Lair 87] and [Lair 93], extends sketches theory. It is somewhat related to notions introduced by Freyd in [Freyd 73] (see [Freyd & Scedrov 90], also).

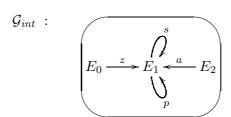
In this paper, a specification is defined as a weft. Wefts generalize algebraic specifications, and the *realizations* of a weft generalize the models of an algebraic specification. Moreover, wefts can be used to define *mosaics*, as will be seen in [guide2], leading to a clear management of states [state], and, more generally, of many implicit features of computer languages.

This paper is made up of three sections, together with an appendix giving some basic notions in category theory.

The aim of section 2 is to define the *wefts*, more precisely the  $\mathcal{A}$ -wefts with respect to an arbitrary category  $\mathcal{A}$  (the definition of a category is given in A.3). We also define the category  $\mathcal{A}mbi$  of ambigraphs, since the  $\mathcal{A}mbi$ -wefts, or wefts of ambigraphs, are quite similar to the algebraic specifications. A weft has realizations, and the more precise a weft is, the less realizations it has.

Here are some examples of specifications, which are presented, in an informal way, from the point of view of wefts. These specifications are more and more precise.

• A directed graph (see A.1 for this well-known notion), made up of points and arrows, is a specification, though a very simple one. A (set-valued) realization of a directed graph is made up of a set for all point of the graph and a map for all arrow. For instance, let us consider the directed graph:



and three realizations, among others, of  $\mathcal{G}_{int}$ . They are called respectively  $\omega_1$ ,  $\omega_2$  and  $\omega_3$ , and they are defined from the set  $\mathbb{Z}$  of integers with the operations *succ* and *pred* (for *successor* and *predecessor*), the addition  $+: \mathbb{Z}^2 \to \mathbb{Z}$  and the constant map  $* \mapsto 0$  from the one-element set  $\{*\}$  to  $\mathbb{Z}$ :

$$\omega_1: \{*\} \xrightarrow{0} \mathbb{Z} \xrightarrow{+} \mathbb{Z}^2 \qquad \omega_2: \mathbb{Z} \xrightarrow{succ} \mathbb{Z} \qquad \omega_3: \{*\} \xrightarrow{0} \mathbb{Z} \xrightarrow{+} \mathbb{Z}^2$$

$$\underset{pred}{\overset{succ}{\longrightarrow}} \omega_3: \{*\} \xrightarrow{0} \mathbb{Z} \xrightarrow{+} \mathbb{Z}^2$$

By composition of arrows from  $\mathcal{G}_{int}$ , we get terms, or programs, which may be interpreted by these realizations. For example the program p(z), composed of the two arrows  $z: E_0 \to E_1$  and  $p: E_1 \to E_1$ , is interpreted in  $\omega_1$  as the integer pred(0) = -1, in  $\omega_2$  as the map  $pred \circ succ: \mathbb{Z} \to \mathbb{Z}$ , i.e. as the identity map of  $\mathbb{Z}$ , and in  $\omega_3$  as the integer succ(0) = 1.

- Adding equations, we get an ambigraph, which is a more precise kind of specification. For instance let us add to  $\mathcal{G}_{int}$  two equations p(s(x)) = x and s(p(x)) = x: we get an ambigraph  $\mathcal{G}'_{int}$ . In a realization of  $\mathcal{G}'_{int}$ , the maps which interpret the arrows s and p are the inverse of each other. Hence  $\omega_1$  and  $\omega_2$  are realizations of  $\mathcal{G}'_{int}$ , but  $\omega_3$  is not. On the other hand, thanks to equations, we can evaluate programs, for example we can replace the program p(s(s(z))) by s(z).
- Now, adding constraints, we get a weft of ambigraphs, which is a still more precise kind of specification. Like an equation, a constraint both restricts the number of realizations and enriches the programming language. For instance, a constraint (let us call it  $\Gamma$ ) allows us to say that whenever  $E_1$  is interpreted as  $\mathbb{Z}$ , then  $E_2$  must be interpreted as  $\mathbb{Z}^2$ . Let  $\mathbf{S}_{int}$  denote the specification made up of the ambigraph  $\mathcal{G}'_{int}$  together with the constraint  $\Gamma$ , then  $\omega_1$  is a realization of  $\mathbf{S}_{int}$ , but  $\omega_2$  is not. On the other hand, this constraint allows us to consider pairs like (s(z), p(z)), hence programs like a(s(z), p(z)). The pair (s(z), p(z)) is interpreted by  $\omega_1$  as the pair of integers (1, -1), and a(s(z), p(z)) as the integer (1) + (-1) = 0.

As a consequence of their definition, wefts are endowed with strong functoriality properties: once something is well defined (in a precise meaning) over ambigraphs, its definition extends automatically to wefts of ambigraphs. It means that the extension of such a definition to the constraints, however powerful they are, does not require any effort.

Section 3 is devoted to projective sketches. Sketch theory has been known since Ehresmann's pioneering work in the 60's. Sketches can be considered as wefts of compositive graphs (compositive graphs, which are defined in A.2, are directed graphs with some composition of arrows) with constraints only for limits (projective or inductive limits, see A.5 and A.6). The set-valued realizations of a sketch are called its models. The restriction on the shape of the constraints results in an easy definition of homomorphisms between models of a given sketch  $\mathbf{E}$ , which gives the category  $\mathcal{M}od(\mathbf{E})$  of models of  $\mathbf{E}$ . Actually, sketches will be used here in order to sketch (or specify) specifications. Indeed, wefts of ambigraphs make up a relevant notion for specification in a purely functional setting. Moreover, in order to take into account implicit features of programming languages, we will use (in [guide2]) wefts with respect to various categories  $\mathcal{A}$ . Each category  $\mathcal{A}$  which we will consider is sketched by a sketch  $\mathbf{E}$ , which means that it is equivalent to the category of models of  $\mathbf{E}$ .

For instance, the category of ambigraphs can be sketched by a projective sketch  $\mathbf{E}_{\mathcal{A}mbi}$ . So, each ambigraph can be identified with a model of  $\mathbf{E}_{\mathcal{A}mbi}$ . Since an ambigraph has points, arrows and equations, the sketch  $\mathbf{E}_{\mathcal{A}mbi}$  has points called Pt, Ar, Eq. The ambigraph  $\mathcal{G}'_{int}$ , for example, can be identified with a model of  $\mathbf{E}_{\mathcal{A}mbi}$ , which interprets the point Pt as the set  $\{E_0, E_1, E_2\}$ , the point Ar as the set  $\{z, s, p, a\}$  and the point Eq as the set  $\{p(s(x)) = x, s(p(x)) = x\}$ .

We will only need *projective sketches*, *i.e.* sketches with constraints for projective limits. From the point of view of their logical power, projective sketches essentially correspond to *positive conditional* (i.e. *Horn-universal*) specifications, while sketches correspond to *first-order* specifications [Guitart & Lair 82]. With wefts, as with patchworks, we can reach *second-order* specifications.

So, from our point of view, an  $\mathcal{A}$ -weft, for any category  $\mathcal{A}$ , is a specification. In particular  $\mathcal{A}$  can be the category  $\mathcal{A}mbi$  of ambigraphs, which is sketched by  $\mathbf{E}_{\mathcal{A}mbi}$ . We will consider other categories  $\mathcal{A}$ , which can also be sketched. Therefore, we need a method for building new sketches, together with a way to recover their categories of models. Such a method is presented in section 4. It is called the *blow-up* of a projective sketch, and it is similar to a well-known construction over categories called the *discrete fibration*. The blow-up generalizes the usual sagittal diagram of a map.

For example, let us consider the two sets  $X = \{x_1, x_2, x_3\}$  and  $X' = \{x'_1, x'_2, x'_3, x'_4\}$  and the map  $f: X \to X'$  such that  $f(x_1) = x'_1$ ,  $f(x_2) = f(x_3) = x'_3$ . The sagittal diagram of f is the directed graph made up of points  $[E, x_i]$  for i from 1 to 3, points  $[E', x'_i]$  for i from 1 to 4, and arrows  $[e, x_i]: [E, x_i] \to [E', f(x_i)]$  for i from 1 to 3:

Now, let us consider the following directed graph  $\mathcal{G}_{\mathcal{M}ap}$ :

$$\mathcal{G}_{\mathcal{M}ap}$$
:  $\left(E \xrightarrow{e} E'\right)$ 

Then f defines a model  $\mathcal{I}_f$  of  $\mathcal{G}_{\mathcal{M}ap}$ :

$$\mathcal{I}_f: X \xrightarrow{f} X'$$

and the blow-up  $\mathcal{G}_{\mathcal{M}ap} \setminus \mathcal{I}_f$  of  $\mathcal{G}_{\mathcal{M}ap}$  by this model is precisely the sagittal diagram of f.

Similarly, the blow-up of  $\mathbf{E}_{\mathcal{A}mbi}$  by an ambigraph  $\mathcal{G}$  counts each point (Pt, Ar, etc.) of  $\mathbf{E}_{\mathcal{A}mbi}$  as many times as the number of elements in the set  $\mathcal{G}(\mathsf{Pt})$ ,  $\mathcal{G}(\mathsf{Ar})$ , etc. For instance, the blow-up of  $\mathbf{E}_{\mathcal{A}mbi}$  by the ambigraph  $\mathcal{G}'_{int}$  replaces the point Pt of  $\mathbf{E}_{\mathcal{A}mbi}$  by three points [Pt,  $E_0$ ], [Pt,  $E_1$ ] and [Pt,  $E_2$ ].

Finally, a fundamental theorem allows us to characterize the models of a projective sketch which is defined as a blow-up.

To sum up, in this paper we define and study two levels of specification:

- 1. the first level is made of wefts, which specify their realizations;
- 2. the second level is made of projective sketches, which sketch their models.

The link between both levels is that we focus on  $\mathcal{A}$ -wefts for a category  $\mathcal{A}$  which is equivalent to the category of models of a projective sketch  $\mathbf{E}$ . In the simplest cases, corresponding to functional programming,  $\mathbf{E}$  is  $\mathbf{E}_{\mathcal{A}mbi}$ , so that  $\mathcal{A} = \mathcal{A}mbi$  is the category of ambigraphs. Other projective sketches, among which blow-ups of  $\mathbf{E}_{\mathcal{A}mbi}$ , will be used in [guide2] and [guide3], in order to define the mosaics and to study their applications to implicit features of programming languages.

No prerequisite is required to read this paper. However, it can prove helpful to be familiar with either algebraic specifications (as introduced in [Goguen et al. 78] or [Astesiano et al. 99] for example) or with categories (part of [Mac Lane 71]) or sketches (as in [Coppey & Lair 84], in [Lellahi 89], or in the first pages of [Duval & Reynaud 94a]).

This paper is intended to be an introductory paper. We motivate our choices from detailed examples, but definitions and results are usually not given in their most general setting, proofs are just... sketched, and technical issues like the *size* of sets are left out. On the other hand, the *reference manual* [ref] gives general definitions and results, comprehensive proofs, and addresses size issues.

#### 2 Wefts

The definition of wefts is given in this section. Examples of wefts for the specification of natural numbers are considered in some detail.

The set of natural numbers can be defined as the set which is freely generated by an element 0 and an operation *succ* (for *successor*). In order to get a weft for the specification of natural numbers, we proceed in the following way.

- First, we consider the directed graph which underlies this definition; for this purpose, we introduce a one-element set  $\mathbb{U} = \{*\}$ , so that the natural number 0 can be seen as the constant map  $* \mapsto 0$  from  $\mathbb{U}$  to  $\mathbb{N}$ . The directed graph is made up of two points U and N (which represent respectively the sets  $\mathbb{U}$  and  $\mathbb{N}$ ) and two arrows  $s: N \to N$  and  $z: U \to N$  (which represent respectively the maps succ and 0). This directed graph is the support of the weft.
- Then, we add two *constraints* to this support. One says that the point *U* represents a one-element set, the other says that we are interested in the set which is freely generated by 0 and *succ*.

In a more general setting, a A-weft (for some category A) is made up of:

- a *support*, which is a point in A;
- and *constraints*, which are defined from points and arrows in A.

In the example above,  $\mathcal{A}$  is the category  $\mathcal{D}ir$  of directed graphs. Often,  $\mathcal{A}$  is the category  $\mathcal{A}mbi$  of ambigraphs, which is more expressive than  $\mathcal{D}ir$ . The wefts of ambigraphs, or  $\mathcal{A}mbi$ -wefts, can be seen as generalizations of the usual algebraic specifications [Goguen et al. 78, Wirsing 90]. Most examples of wefts in this paper are either wefts of ambigraphs or wefts of compositive graphs. However in [guide2] we will focus on  $\mathcal{A}$ -wefts for other categories  $\mathcal{A}$ . For this reason, in this section:

#### $\mathcal{A}$ is any category.

In 2.1 we define the category  $\mathcal{A}mbi$  of ambigraphs: an ambigraph is just a compositive graph (see A.2) plus equations. Constraints with respect to any category are defined in 2.3, and a recursive definition of  $\mathcal{A}$ -wefts follows in 2.4. The comprehensive definition of wefts will be given in the reference manual, here we restrict our definition to rather simple constraints. Actually it is the definition of the realizations of a  $\mathcal{A}$ -weft, as given in 2.5, which enlightens the meaning of the constraints. Beforehand, one constraint is studied in some detail in 2.2.

#### 2.1 Ambigraphs

Ambigraphs are compositive graphs (the definition of compositive graphs is given in A.2) with equations, which are just pairs of arrows with the same rank.

Ambifunctors between ambigraphs are functors between the sublying compositive graphs which preserve equations. It follows that ambifunctors from an ambigraph to a category map each equation to an equality.

**Definition 1** An ambigraph  $\mathcal{G}$  is a compositive graph together with:

• a set of equations  $\mathcal{G}(Eq) \subseteq \mathcal{G}(RankP)$ , whose elements are denoted  $g_l \equiv g_r$  rather than  $(g_l, g_r)$ .

Let  $\mathcal{G}$  and  $\mathcal{G}'$  be ambigraphs. An ambifunctor  $\varphi: \mathcal{G} \to \mathcal{G}'$  is a functor between the sublying compositive graphs such that:

• for all equation  $g_l \equiv g_r$  of  $\mathcal{G}$ , the pair of arrows (with the same rank)  $(\varphi(g_l), \varphi(g_r))$  is an equation  $\varphi(g_l) \equiv \varphi(g_l)$  of  $\mathcal{G}'$ .

An ambifunctor  $\varphi: \mathcal{G} \to \mathcal{G}'$  is called an *extension* if it is an inclusion both on points and on arrows. Then it is also an inclusion on identity arrows, on composable pairs and on equations. We say that  $\mathcal{G}'$  extends  $\mathcal{G}$ .

Ambigraphs with ambifunctors are the points and arrows of a category:

Ambi.

Moreover, let us consider a category of sets:

Set,

restricted to small sets, see [ref1]. Its points are called the sets and its arrows the maps.

Clearly each category  $\mathcal{V}$ , with equations  $v \equiv v$  for all the arrows v in  $\mathcal{V}$ , becomes an ambigraph. It follows that an ambifunctor from an ambigraph  $\mathcal{G}$  towards a category  $\mathcal{V}$  maps each equation of  $\mathcal{G}$  to an equality between arrows of  $\mathcal{V}$ .

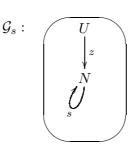
**Example 1** We now build, from the natural numbers, two ambigraphs  $\mathcal{G}_s$  and  $\mathcal{G}_{s,p}$ , and two ambifunctors from  $\mathcal{G}_s$  and  $\mathcal{G}_{s,p}$  respectively towards the category  $\mathcal{S}et$ .

The ambigraph  $\mathcal{G}_s$  has no identity arrow, no composable pair of arrows, and no equation:

Ambigraph  $\mathcal{G}_s$ :

points: U, N,

arrows:  $z: U \to N, s: N \to N$ .



Let us consider the set  $\mathbb{N}$  of natural numbers, with the element 0 and the map successor succ(n) = n + 1. Let  $\mathbb{U} = \{*\}$  be a one-element set; the constant map  $* \mapsto 0 : \mathbb{U} \to \mathbb{N}$  is also denoted 0. The ambifunctor  $\mathcal{N}_s$  from  $\mathcal{G}_s$  to  $\mathcal{S}et$  is defined as follows:

Ambifunctor  $\mathcal{N}_s:\mathcal{G}_s\to\mathcal{S}et$ :

points:  $N \mapsto \mathbb{N}, U \mapsto \mathbb{U},$ arrows:  $s \mapsto succ, z \mapsto 0.$ 



The ambigraph  $\mathcal{G}_{s,p}$  extends  $\mathcal{G}_s$ :

Ambigraph  $\mathcal{G}_{s,p}$ :

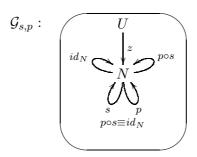
extends:  $\mathcal{G}_s$ ,

arrows:  $p: N \to N, id_N: N \to N, p \circ s: N \to N,$ 

identity arrow:  $id_N: N \to N$ ,

composable pair: (s, p),

 $\begin{array}{ll} composed \ arrow \colon & p \circ s \colon N \to N, \\ equation \colon & p \circ s \equiv id_N. \end{array}$ 

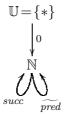


The map  $predecessor\ \widetilde{pred}: \mathbb{N} \to \mathbb{N}$  is defined as  $\widetilde{pred}(n) = n-1$  if  $n \neq 0$  and  $\widetilde{pred}(0) = 0$ . The ambifonctor  $\mathcal{N}_{s,p}$  from  $\mathcal{G}_{s,p}$  to  $\mathcal{S}et$  is defined as follows:

Ambifunctor  $\mathcal{N}_{s,p}:\mathcal{G}_{s,p}\to\mathcal{S}et$ :

extends:  $\mathcal{N}_s$ ,

arrow:  $p \mapsto \widetilde{pred}$ .



It is an ambifunctor since it maps the equation  $p \circ s \equiv id_N$  on the equality between maps  $\widetilde{pred} \circ succ = id_N$ , i.e. the equality:

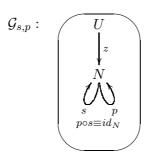
$$\forall n \in \mathbb{N}, \ \widetilde{pred}(succ(n)) = n.$$

In the description of the ambigraph  $\mathcal{G}_{s,p}$ , for the equation  $p \circ s \equiv id_N$  to make sense, it is necessary that  $\mathcal{G}_{s,p}$  contains the identity arrow  $id_N$ , the composable pair (s,p) and its composed arrow  $p \circ s$ . This is why we are allowed to use the following abbreviated description:

Ambigraph  $\mathcal{G}_{s,p}$ :

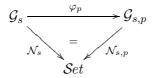
extends:  $\mathcal{G}_s$ , arrow:  $p: N \to N$ ,

equation:  $p \circ s \equiv id_N$ .



From now on, such abbreviated descriptions will be used for ambigraphs.

Let  $\varphi_p: \mathcal{G}_s \to \mathcal{G}_{s,p}$  be the extension functor. Then clearly  $\mathcal{N}_s = \mathcal{N}_{s,p} \circ \varphi_p$ :



It is worth noticing that the image of the point U by an ambifunctor from  $\mathcal{G}_s$  or  $\mathcal{G}_{s,p}$  to  $\mathcal{S}et$  can be any set, not just a one-element set. We are not yet able to lay down such a property, however it will become possible later, thanks to realizations of wefts: see example 5.

In order to justify the use of equations, *i.e.* the use of ambigraphs rather than compositive graphs, let us come back to example 1.

**Example 2** In example 1, the equation  $(p \circ s \equiv id_N)$  can be avoided, by replacing the ambigraph  $\mathcal{G}_{s,p}$  by the following compositive graph:

Compositive graph  $\mathcal{G}'_{s,p}$ :

points: U, N,

arrows:  $z:U\to N,\, s:N\to N,\, p:N\to N,\, id_N:N\to N,$ 

identity arrow:  $id_N: N \to N$ ,

composable pair: (s, p),

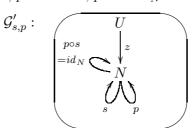
composed arrow:  $p \circ s = id_N : N \to N$ .

or, with the abbreviated description:

Compositive graph  $\mathcal{G}'_{s,p}$ :

points: U, N,

arrows:  $z: U \to N, s: N \to N, p: N \to N, p \circ s = id_N: N \to N.$ 



The ambifunctor  $\mathcal{N}_{s,p}:\mathcal{G}_{s,p}\to\mathcal{S}et$  can be replaced by the following functor:

Functor  $\mathcal{N}'_{s,p}: \mathcal{G}'_{s,p} \to \mathcal{S}et:$ points:  $N \mapsto \mathbb{N}, U \mapsto \mathbb{U},$ arrows:  $s \mapsto succ, z \mapsto 0, p \mapsto \widetilde{pred}.$ 

One of the reasons for choosing ambigraphs rather than compositive graphs is well-known, and plays a crucial part in [Duval & Reynaud 94a, Duval & Reynaud 94b] (where the symbol  $\sim$  is used instead of  $\equiv$ ): in the compositive graph  $\mathcal{G}'_{s,p}$  we cannot describe a computation process which replaces  $p \circ s$  by  $id_N$ , since both are already equal... A second reason for choosing ambigraphs rather than compositive graphs will become clear in [guide2] and [guide3], where we will have to interpret the equations as a relation weaker than equality.

#### 2.2 An example of a constraint

Let  $\omega$  be an ambifunctor from  $\mathcal{G}_s$  to  $\mathcal{S}et$ . How is it possible to ensure that the set  $\omega(U)$  has exactly one element? This property  $\wp_T$ :

•  $\omega(U)$  is a one-element set

is satisfied by some ambifunctors from  $\mathcal{G}_s$  to  $\mathcal{S}et$ , like  $\mathcal{N}_s$ , but not by all of them. Actually, the property  $\wp_T$  can be associated with something called a *constraint*, and denoted  $\Gamma_T$ , in such a way that the ambifunctors from  $\mathcal{G}_s$  to  $\mathcal{S}et$  which satisfy the property  $\wp_T$  are exactly those which satisfy the constraint  $\Gamma_T$ . Constraints are defined in 2.3 and satisfaction in 2.5. Now, let us look more closely at the constraint  $\Gamma_T$ .

First, let us give an abstract version of the property  $\wp_T$ . The one-element sets are exactly the *terminal* sets, in the following sense: a set  $\mathbb{U}$  is *terminal* if for all set X there exists a unique map from X to  $\mathbb{U}$ :

$$\forall X$$

$$\exists ! f \downarrow$$

$$\exists \exists f \downarrow$$

Hence the property  $\wp_T$  can be stated as:

•  $\omega(U)$  is a terminal set.

It follows that the constraint  $\Gamma_T$  must be such that an ambifunctor  $\omega : \mathcal{G}_s \to \mathcal{S}et$  satisfies  $\Gamma_T$  if and only if the set  $\omega(U)$  (which we denote  $\mathbb{U}$ ) is terminal.

- First, we must pick out the point U in  $\mathcal{G}_s$ , since it is the point on which the constraint acts. For this purpose, we introduce the ambigraph  $\mathcal{C}_T$  made up of a single point, and the ambifunctor  $\chi: \mathcal{C}_T \to \mathcal{G}_s$  such that  $\chi(C) = U$ . The ambifunctor  $\omega_{\mathcal{C}} = \omega \circ \chi: \mathcal{C}_T \to \mathcal{S}et$  is characterized by  $\omega_{\mathcal{C}}(C) = \mathbb{U}$ .
- Now, in order to express the terminality property, we must be able to pick out pairs of sets  $(\mathbb{U}, X)$  for any set X. For this purpose, we introduce the ambigraph  $\mathcal{D}_T$  made up of two points C and D, and the extension ambifunctor  $\gamma_T : \mathcal{C}_T \to \mathcal{D}_T$ . Then the pairs of sets  $(\mathbb{U}, X)$  are the pairs  $(\omega_{\mathcal{D}}(C), \omega_{\mathcal{D}}(D))$  where  $\omega_{\mathcal{D}} : \mathcal{D}_T \to \mathcal{S}et$  are the ambifunctors which extend  $\omega_{\mathcal{C}}$ , in the sense that  $\omega_{\mathcal{D}} \circ \gamma_T = \omega_{\mathcal{C}}$ .

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• Lastly, we introduce the ambigraph  $\mathcal{U}_T$  made up of two points C and D and one arrow  $u:D\to C$ , and the extension ambifunctor  $\delta_T:\mathcal{D}_T\to\mathcal{U}_T$ . Let  $\omega_{\mathcal{D}}$  be as above, and let  $X=\omega_{\mathcal{D}}(D)$ . The maps  $f:X\to\mathbb{U}$  are the maps  $\omega_{\mathcal{U}}(u):\omega_{\mathcal{U}}(D)\to\omega_{\mathcal{U}}(C)$  where  $\omega_{\mathcal{U}}:\mathcal{U}_T\to\mathcal{S}et$  are the ambifunctors which extend  $\omega_{\mathcal{D}}$ , in the sense that  $\omega_{\mathcal{U}}\circ\delta_T=\omega_{\mathcal{D}}$ .

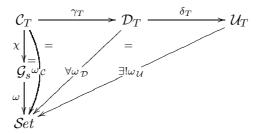
$$\mathcal{C}_T:$$
 $C$ 
 $C$ 
 $\mathcal{D}_T:$ 
 $C$ 
 $C$ 
 $\mathcal{D}_T:$ 
 $C$ 
 $\mathcal{D}_T:$ 
 $C$ 
 $\mathcal{D}_T:$ 
 $\mathcal{D}_T:$ 

The constraint  $\Gamma_T$  is made up of the ambigraphs  $\mathcal{C}_T$ ,  $\mathcal{D}_T$  and  $\mathcal{U}_T$  and the ambifunctors  $\chi$ ,  $\gamma_T$  and  $\delta_T$ :

$$\begin{array}{ccc}
\mathcal{C}_T & \xrightarrow{\gamma_T} & \mathcal{D}_T & \xrightarrow{\delta_T} & \mathcal{U}_T \\
\chi & & & & \\
\mathcal{G}_s & & & & \\
\end{array}$$

An ambifunctor  $\omega: \mathcal{G}_s \to \mathcal{S}et$  satisfies  $\Gamma_T$  if and only if, with  $\omega_{\mathcal{C}} = \omega \circ \chi: \mathcal{C}_T \to \mathcal{S}et$ :

• for all ambifunctor  $\omega_{\mathcal{D}}$  from  $\mathcal{D}_T$  to Set such that  $\omega_{\mathcal{D}} \circ \gamma_T = \omega_{\mathcal{C}}$ , there exists a unique ambifunctor  $\omega_{\mathcal{U}}$  from  $\mathcal{U}_T$  to Set such that  $\omega_{\mathcal{U}} \circ \delta_T = \omega_{\mathcal{D}}$ .



Henceforth,  $\omega$  satisfies  $\Gamma_T$  if and only if  $\omega(U)$  is a terminal set, i.e. a one-element set, as required.

$$\forall X = \omega_{\mathcal{D}}(D) = \omega_{\mathcal{U}}(D)$$

$$\exists ! f = \omega_{\mathcal{U}}(u) \Big|$$

$$\omega(U) = \omega_{\mathcal{C}}(C) = \omega_{\mathcal{D}}(C) = \omega_{\mathcal{U}}(C)$$

In the following, we define  $\mathcal{A}$ -wefts and their realizations in such a way that, if we add to  $\mathcal{G}_s$  this constraint  $\Gamma_T$ , we get an  $\mathcal{A}mbi$ -weft  $\mathbf{S}_s$  (example 5) whose set-valued realizations are the ambifunctors  $\omega$  from  $\mathcal{G}_s$  to  $\mathcal{S}et$  such that  $\omega(U)$  is a one-element set. As a consequence, the ambifunctor  $\mathcal{N}_s$  from  $\mathcal{G}_s$  to  $\mathcal{S}et$  is a set-valued realization of  $\mathbf{S}_s$ .

#### 2.3 Constraints

Let us consider a category W and a functor  $Supp : W \to A$ . For example W can be the category A itself and Supp the identity. Later, W will be a category of A-wefts and Supp will be the support functor.

**Definition 2** Let W be a category,  $Supp : W \to A$  a functor and  $A_0$  a point of A. A constraint  $\Gamma$  over  $A_0$  relative to  $Supp : W \to A$  is made up of:

• three points C, D and U of W,

- a pair of consecutive arrows  $\mathbf{C} \xrightarrow{\gamma} \mathbf{D} \xrightarrow{\delta} \mathbf{U}$  of  $\mathcal{W}$ ,
- and an arrow  $Supp(\mathbf{C}) \xrightarrow{\chi} A_0$  of  $\mathcal{A}$ .

The arrow  $\chi$  is the body of the constraint, and the pair of consecutive arrows  $(\gamma, \delta)$  is its potential.

In such a constraint  $(\chi, (\gamma, \delta))$ , the body  $\chi$  is used to pick out in  $A_0$  the image of  $Supp(\mathbf{C})$  by  $\chi$ . The potential  $(\gamma, \delta)$  is used to lay down some property of the image of  $\chi(Supp(\mathbf{C}))$  by an arrow  $\omega : A_0 \to A$  (where A is a point of A): this property is made precise in 2.5 through the notion of satisfaction.

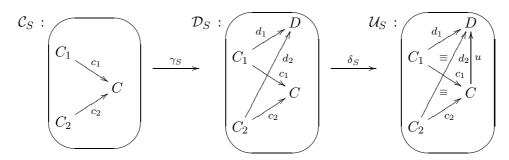
**Example 3** Let both  $\mathcal{A}$  and  $\mathcal{W}$  be the category of ambigraphs, and let Supp be the identity. Then  $A_0$  is an ambigraph, called  $\mathcal{G}$ . In the same way  $\mathbf{C}$ ,  $\mathbf{D}$  and  $\mathbf{U}$  are ambigraphs, called  $\mathcal{C}$ ,  $\mathcal{D}$  and  $\mathcal{U}$ , and  $\chi$ ,  $\gamma$  and  $\delta$  are ambifunctors. For instance, a *terminal point* constraint has potential:

$$\mathcal{C}_T \xrightarrow{\gamma_T} \mathcal{D}_T \xrightarrow{\delta_T} \mathcal{U}_T$$

as described in 2.2. Let  $G = \chi(C)$  be the image of the point C of  $\mathcal{C}_T$  in  $\mathcal{G}$ . This is usually denoted:

$$G=11$$
.

**Example 4** Here also let  $\mathcal{A} = \mathcal{W} = \mathcal{A}mbi$  and let Supp be the identity. A binary sum constraint over an ambigraph  $\mathcal{G}$  has potential:



Let  $(G_1 \xrightarrow{g_1} G \xleftarrow{g_2} G_2) = \chi(C_1 \xrightarrow{c_1} C \xleftarrow{c_2} C_2)$  be the image of  $C_S$  in G. Usually, when  $G_1$  and  $G_2$  are clear from the context, this is denoted:

$$G = G_1 + G_2$$
.

#### 2.4 Wefts and weft homomorphisms

Each  $\mathcal{A}$ -weft has a level, which is a non-negative integer, and a support, which is a point of  $\mathcal{A}$ . An  $\mathcal{A}$ -weft of level 0 is just made up of its support: it is a point of  $\mathcal{A}$ . An  $\mathcal{A}$ -weft of level  $\leq n$  (for some integer  $n \geq 1$ ) is made up of a support and constraints, which themselves are made up of  $\mathcal{A}$ -weft and  $\mathcal{A}$ -weft homomorphisms of level  $\leq n-1$ .

**Definition 3** The category  $Weft_0(A)$  of A-wefts of level 0 is equal to A. The functor  $Supp_0$ :  $Weft_0(A) \to A$  is the identity.

For all integer  $n \ge 1$ , assume that we know the category  $Weft_{n-1}(\mathcal{A})$  of  $\mathcal{A}$ -wefts of level  $\le n-1$ , as well as the functor  $Supp_{n-1}: Weft_{n-1}(\mathcal{A}) \to \mathcal{A}$ .

An  $\mathcal{A}$ -weft  $\mathbf{S}$  of level  $\leq n$  is made up of a point  $Supp_n(\mathbf{S})$  of  $\mathcal{A}$ , called the support of  $\mathbf{S}$ , and a set of constraints over  $Supp_n(\mathbf{S})$  in the category  $Weft_{n-1}(\mathcal{A})$ , called the constraints of  $\mathbf{S}$ . An

 $\mathcal{A}\text{-}weft\ homomorphism}\ \sigma: \mathbf{S} \to \mathbf{S}' \text{ of level} \leq n \text{ is an arrow } Supp_n(\sigma) \text{ from } Supp_n(\mathbf{S}) \text{ to } Supp_n(\mathbf{S}') \text{ in } \mathcal{A}, \text{ such that for all constraint } (Supp_{n-1}(\mathbf{C}) \xrightarrow{\chi} Supp_n(\mathbf{S}), \mathbf{C} \xrightarrow{\gamma} \mathbf{D} \xrightarrow{\delta} \mathbf{U}) \text{ of } \mathbf{S}, \text{ the body:}$ 

$$Supp_n(\sigma) \circ \chi : Supp_{n-1}(\mathbf{C}) \to Supp_n(\mathbf{S}')$$

together with the same potential:

$$\mathbf{C} \xrightarrow{\gamma} \mathbf{D} \xrightarrow{\delta} \mathbf{U}$$

is a constraint of S'. In this way we get the category  $Weft_n(\mathcal{A})$  of  $\mathcal{A}$ -wefts of level  $\leq n$ , with the support functor  $Supp_n : Weft_n(\mathcal{A}) \to \mathcal{A}$ .

It is easy to check that  $Weft_{n-1}(A)$  is a subcategory of  $Weft_n(A)$ , hence A is a subcategory of  $Weft_n(A)$ . The union, or more exactly the *inductive limit*, as defined in A.6, of the categories  $Weft_n(A)$  (for  $n \in \mathbb{N}$ ) is the category of A-wefts:

$$Weft(A)$$
.

Since the support functors  $Supp_n$  are compatible, they define a functor:

$$Supp: Weft(A) \to A$$
.

Another notation is used for supports: for an  $\mathcal{A}$ -weft  $\mathbf{S}$  we write  $\underline{\mathbf{S}}$  for  $Supp(\mathbf{S})$ ; and for an  $\mathcal{A}$ -weft homomorphism  $\sigma$  we write  $\underline{\sigma}$ , or just  $\sigma$ , for  $Supp(\sigma)$ .

Since  $\underline{\mathbf{S}}$  is a point of  $\mathcal{A}$ , it is also an  $\mathcal{A}$ -weft (of level 0). The fact of adding constraints to  $\underline{\mathbf{S}}$ , in order to get the  $\mathcal{A}$ -weft  $\mathbf{S}$ , is an  $\mathcal{A}$ -weft homomorphism:

$$\underline{\mathbf{S}} \hookrightarrow \mathbf{S}$$

When S has exactly one constraint:

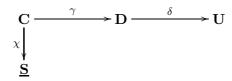
$$(\underline{\mathbf{C}} \xrightarrow{\chi} \underline{\mathbf{S}}, \mathbf{C} \xrightarrow{\gamma} \mathbf{D} \xrightarrow{\delta} \mathbf{U})$$

it can be represented as follows:

$$\begin{array}{c}
\underline{\mathbf{C}} & \hookrightarrow & \mathbf{C} & \xrightarrow{\gamma} & \mathbf{D} & \xrightarrow{\delta} & \mathbf{U} \\
\chi \downarrow & & & \\
\underline{\mathbf{S}} & & & & \\
\end{array}$$

This drawing represents both the potential  $\mathbf{C} \xrightarrow{\gamma} \mathbf{D} \xrightarrow{\delta} \mathbf{U}$  (in  $\mathcal{W}eft(\mathcal{A})$ ) and the body  $\underline{\mathbf{C}} \xrightarrow{\chi} \underline{\mathbf{S}}$  (in  $\mathcal{A}$ ), which are related through the  $\mathcal{A}$ -weft homomorphism  $\underline{\mathbf{C}} \xrightarrow{\zeta} \mathbf{C}$  (in  $\mathcal{W}eft(\mathcal{A})$ ).

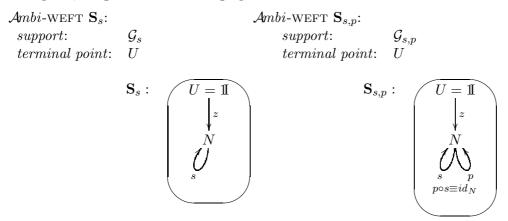
When the potential has level 0, then  $\underline{\mathbf{C}} = \mathbf{C}$  and the whole diagram is in the category  $\mathcal{A}$ . In this case it is possible to simplify the drawing (as was done in 2.2):



More generally, for any level of the potential, this drawing can be used recursively to get a diagram which is entirely in the category  $\mathcal{A}$ , see example 6. The word weft derives from these diagrams.

Of course, it is easy to generalize these drawings to several constraints.

**Example 5** By adding the constraint  $U = \mathbb{I}$  (of level 0) to the ambigraph  $\mathcal{G}_s$  (resp.  $\mathcal{G}_{s,p}$ ) in example 1, we get a weft of ambigraphs of level 1.



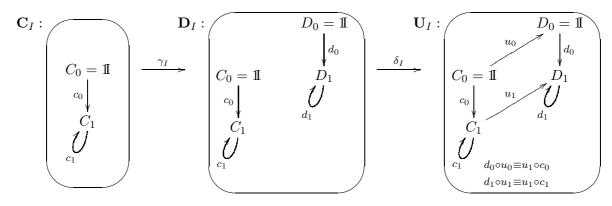
Both these examples correspond to the following diagram in  $\mathcal{A}mbi$ :

$$\begin{array}{ccc}
\mathcal{C}_T & \xrightarrow{\gamma_T} & \mathcal{D}_T & \xrightarrow{\delta_T} & \mathcal{U}_T \\
C \mapsto U & & & \\
\mathcal{G}_s \text{ or } \mathcal{G}_{s,p} & & & \\
\end{array}$$

**Example 6** Let us consider once more the weft of ambigraphs  $\mathbf{S}_s$  from example 5: it allows us to characterize (in a way that will be made precise below, by the definition of realizations) the sets which are endowed with an element and a unary operation. However it is far from enough to characterize the set of natural numbers; we must add the information that it is freely generated by 0 and succ. This amounts to saying that it satisfies the following initiality property:

• for all set X endowed with an element  $x_0 \in X$  and a map  $f: X \to X$  there exists a unique map  $u: \mathbb{N} \to X$  such that  $u(0) = x_0$  and f(u(n)) = u(s(n)) for all  $n \in \mathbb{N}$ .

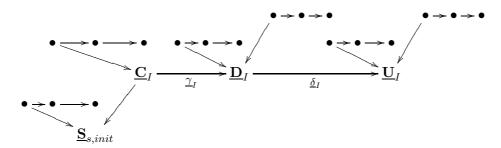
Let us build a weft of ambigraphs  $\mathbf{S}_{s,init}$  by adding to  $\mathbf{S}_s$  an *initiality* constraint  $\Gamma_I$ . The potential of this constraint is the following:



and its body is defined as:

Ambifunctor  $\chi_I : \underline{\mathbf{C}}_I \to \mathcal{G}_s$ : points:  $C_0 \mapsto U$ ,  $C_1 \mapsto N$ , arrows:  $c_0 \mapsto z$ ,  $c_1 \mapsto s$ . 2 WEFTS 14

The wefts of ambigraphs  $\mathbf{C}_I$ ,  $\mathbf{D}_I$  and  $\mathbf{U}_I$  have level 1 because they have constraints of level 0. Henceforth the weft of ambigraphs  $\mathbf{S}_{s,init}$  has level 2. This example corresponds to the following diagram in  $\mathcal{A}mbi$ , where each line  $(\bullet \to \bullet \to \bullet)$  represents  $(\mathcal{C}_T \xrightarrow{\gamma_T} \mathcal{D}_T \xrightarrow{\delta_T} \mathcal{U}_T)$ :



Wefts of ambigraphs generalize algebraic specifications. However, example 6 shows that the initiality constraint over natural numbers can be handled by the wefts in the same way as the terminality constraint over the point U. This point of view on constraints is highly powerful and plays a fundamental role. It is borrowed from [Lair 87] and [Lair 93] (see also the notion of canvas in [Ageron 91]). On the contrary, algebraic specifications use fairly different techniques in order to handle these two constraints, see [Wirsing 90].

**Example 7** The following weft of ambigraphs of level 1 will be used [guide2] in order to study error handling. It includes a terminal point constraint and a binary sum constraint, as described in examples 3 and 4.

 $\mathcal{A}mbi$ -WEFT  $\mathbf{S}_e$ :

points:  $H, H', H^e$ 

arrows:  $h: H \to H', h^e: H^e \to H'$ 

terminal point:  $H^e$ 

 $sum: \hspace{1cm} H \xrightarrow{h} H' \xleftarrow{h^e} H^e$ 

$$\mathbf{S}_e: \quad \overbrace{H \xrightarrow{h} H' = H + H^e \xleftarrow{h^e} H^e = \mathbb{1}}$$

#### 2.5 Realizations of a weft

It is now, with the definition of the *realizations* of an  $\mathcal{A}$ -weft  $\mathbf{S}$ , that we explicitly give the property which is associated with the potential of a constraint. A *realization* of  $\mathbf{S}$  is defined as an arrow in  $\mathcal{A}$  with domain the support of  $\mathbf{S}$  which *satisfies the constraints* of  $\mathbf{S}$ ; satisfaction is defined recursively on the level of the weft (the realizations of a weft are called its *models* in [Lair 87]).

**Definition 4** Let **S** be an  $\mathcal{A}$ -weft and A a point of  $\mathcal{A}$ . If **S** has level 0, a *realization* of **S** towards A is an arrow  $\omega : \underline{\mathbf{S}} \to A$  in  $\mathcal{A}$ .

Let n be an integer  $\geq 1$ . Assume that:

- **S** has level  $\leq n$ ;
- for all A-weft S' of level  $\leq n-1$ , we know the set  $\text{Real}_{A,n-1}(S',A)$  of realizations of S' towards A;

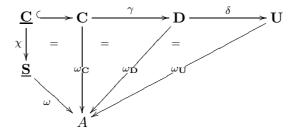
• and  $\operatorname{Real}_{A,n-1}(\mathbf{S}',A)$  is a subset of the set  $\operatorname{Hom}_{\mathcal{A}}(\underline{\mathbf{S}}',A)$  of arrows from  $\underline{\mathbf{S}}'$  to A in the category  $\mathcal{A}$ .

Let:

$$\Gamma = (\underline{\mathbf{C}} \xrightarrow{\chi} \underline{\mathbf{S}}, \ \mathbf{C} \xrightarrow{\gamma} \mathbf{D} \xrightarrow{\delta} \mathbf{U})$$

be a constraint of **S**. Then an arrow  $\omega : \underline{\mathbf{S}} \to A$  of  $\mathcal{A}$  satisfies the constraint  $\Gamma$  if:

- $\omega \circ \chi : \mathbf{C} \to A$  is a realization  $\omega_{\mathbf{C}}$  of  $\mathbf{C}$  towards A;
- and for all realization  $\omega_{\mathbf{D}}$  of  $\mathbf{D}$  towards A such that  $\omega_{\mathbf{D}} \circ \gamma = \omega_{\mathbf{C}}$ , there exists a unique realization  $\omega_{\mathbf{U}}$  of  $\mathbf{U}$  towards A such that  $\omega_{\mathbf{U}} \circ \delta = \omega_{\mathbf{D}}$ .



A realization of **S** towards A is an arrow from  $\underline{\mathbf{S}}$  towards A which satisfies every constraint of **S**.

Then, as usual, we write:

$$\omega: \mathbf{S} \to A$$
,

though **S** and A belong to two distinct categories: A is a point of  $\mathcal{A}$ , whereas **S** is a point of  $\mathcal{W}eft(\mathcal{A})$ . Although the category  $\mathcal{A}$  can be identified to the subcategory  $\mathcal{W}eft_0(\mathcal{A})$  of  $\mathcal{W}eft(\mathcal{A})$ , a realization of **S** towards A is not an arrow in  $\mathcal{W}eft(\mathcal{A})$  (except when **S** has level 0).

It is easy to check the compatibility of realizations at various levels, and to define the set:

$$\operatorname{Real}_{\mathcal{A}}(\mathbf{S}, A)$$

of realizations (of any level) of **S** towards A. Let  $\sigma : \mathbf{S} \to \mathbf{S}'$  be an A-weft homomorphism. Then we define:

$$\operatorname{Real}_{\mathcal{A}}(\sigma, A) : \operatorname{Real}_{\mathcal{A}}(\mathbf{S}', A) \to \operatorname{Real}_{\mathcal{A}}(\mathbf{S}, A)$$

as the map which assigns to each realization  $\omega'$  of  $\mathbf{S}'$  towards A the realization  $\omega' \circ \sigma$  of  $\mathbf{S}$  towards A. In this way we get a contravariant functor:

$$\operatorname{Real}_{\mathcal{A}}(-,A): \mathcal{W}eft(\mathcal{A}) \longrightarrow \mathcal{S}et$$
.

Let **S** be a weft of ambigraphs. The elements of Real<sub>Ambi</sub>(**S**, Set), i.e. the realizations of **S** towards Set, are called the set-valued realizations of **S**.

If **S** is an  $\mathcal{A}$ -weft, A a point of  $\mathcal{A}$  and  $\omega$  a realization of **S** towards A, then we say that **S** specifies  $\omega$  (with respect to  $\mathcal{A}$  and A). Actually such a good notion of realizations is half what is needed for a specification tool. The second half is a good formalization of programs. It will be seen in [guide3] that indeed  $\mathcal{A}$ -wefts can be used to describe functional programming.

**Example 8** Let **S** be a weft of ambigraphs with a terminal point constraint  $\chi : \mathcal{C}_T \to \underline{\mathbf{S}}$  (see the example 3). Let  $\omega$  be a realization of **S** towards a category  $\mathcal{V}$ . From the definition above:

• for all point V of V, there exists a unique arrow  $f: V \to \omega(\chi(C))$  in V.

This means that  $\omega(\chi(C))$  is a terminal point of  $\mathcal{V}$ . When  $\mathcal{V}$  is the category of sets, it means that  $\omega(\chi(C))$  is a one-element set.

For instance, the ambifunctor  $\mathcal{N}_s: \mathcal{G}_s \to \mathcal{S}et$  (example 1) defines a set-valued realization of  $\mathbf{S}_s$  (example 5), since  $\mathcal{N}_s(U) = \{*\}$ . In the same way, the ambifunctor  $\mathcal{N}_{s,p}: \mathcal{G}_{s,p} \to \mathcal{S}et$  (example 1) defines a set-valued realization of  $\mathbf{S}_{s,p}$  (example 5). Note that  $\mathbf{S}_s$  and  $\mathbf{S}_{s,p}$  have many other set-valued realizations.

**Example 9** The weft of ambigraphs  $\mathbf{S}_{s,init}$  in example 6 is made up of  $\mathbf{S}_s$  and the constraint  $\Gamma_I$ . It follows that the realizations of  $\mathbf{S}_{s,init}$  are the realizations of  $\mathbf{S}_s$  which satisfy the constraint  $\Gamma_I$ . Let  $\omega: \mathbf{S}_s \to \mathcal{S}et$  be a set-valued realization of  $\mathbf{S}_s$ ; it is characterized by a set  $\omega(N)$  with an element  $\omega(z) \in \omega(N)$  and an operation  $\omega(s): \omega(N) \to \omega(N)$ . Then  $\omega$  defines a set-valued realization of  $\mathbf{S}_{s,init}$  if and only if:

- $\omega \circ \chi_I : \underline{\mathbf{C}}_I \to \mathcal{S}et$  is a set-valued realization of  $\mathbf{C}_I$ : it means that  $\omega(U)$  is terminal, which is already true since  $\omega$  satisfies the constraint  $U = \mathbb{I}$  of  $\mathbf{S}_s$ ;
- and for all set X with an element  $x_0 \in X$  and a map  $f: X \to X$ , there exists a unique map  $u: \omega(N) \to X$  such that  $u \circ \omega(z) = x_0$  and  $f \circ u = u \circ \omega(s)$ .

It is easy to check that  $\mathcal{N}_s: \mathcal{G}_s \to \mathcal{S}et$  defines a set-valued realization of  $\mathbf{S}_{s,init}$ , and that it is the only one up to isomorphism. Hence,  $\mathbf{S}_{s,init}$  specifies *precisely* the natural numbers.

**Example 10** Let **S** be a weft of ambigraphs with a sum constraint  $\chi : \mathcal{C}_S \to \underline{\mathbf{S}}$  (example 4). Let  $\omega$  be a set-valued realization of **S**. It follows from the definition above that, up to isomorphism:

• the set  $\omega(\chi(C))$  is the disjoint union of  $\omega(\chi(C_1))$  and  $\omega(\chi(C_2))$ , and the maps  $\omega(\chi(c_1))$  and  $\omega(\chi(c_2))$  are the inclusions.

So, a set-valued realization  $\omega$  of the weft of ambigraphs  $\mathbf{S}_e$  from example 7 is characterized, up to isomorphism, by a non-empty set  $\omega(H')$  with a special element  $\varepsilon$ , image of the one-element set  $\omega(H^e)$  by the map  $\omega(h^e)$ . Then the image of the set  $\omega(H)$  by the map  $\omega(h)$  is the complement of  $\{\varepsilon\}$  in  $\omega(H')$ . In other words, if  $\square$  denotes the disjoint union,  $\omega(H')$  is equal to  $\omega(H) \square \{\varepsilon\}$ :

$$\omega(\mathbf{S}_e): \quad \left(\omega(H) \xrightarrow{\omega(h)} \omega(H') = \omega(H) \sqcup \{\varepsilon\} \xrightarrow{\omega(h^e)} \{\varepsilon\}\right)$$

In general, given an  $\mathcal{A}$ -weft  $\mathbf{S}$  and a point A of  $\mathcal{A}$ , it is possible to define many notions of homomorphisms between the realizations of  $\mathbf{S}$  towards A, among which none is more canonical than the others: see [Lair 93]. However, in some special and important cases, there is a natural definition for such homomorphisms: we will see in 3 that this is the case when  $\mathbf{S}$  is a projective sketch (this remains true when  $\mathbf{S}$  is any sketch).

#### 2.6 Functoriality of wefts

The definition of  $\mathcal{A}$ -wefts is such that any functor on the category  $\mathcal{A}$  gives rise to a functor on the category  $\mathcal{W}eft(\mathcal{A})$  of  $\mathcal{A}$ -wefts. For example, as soon as a functor is defined on ambigraphs, we get a functor on wefts of ambigraphs. Indeed the construction of  $\mathcal{A}$ -wefts is functorial in  $\mathcal{A}$ . This property will be used in [guide2] in order to define the crown product for wefts from the crown product for their supports. Precisely:

Let  $\Phi: \mathcal{A} \to \mathcal{A}'$  be a functor between two categories  $\mathcal{A}$  and  $\mathcal{A}'$ . Then  $\Phi$  determines a functor  $Weft(\Phi): Weft(\mathcal{A}) \to Weft(\mathcal{A}')$  which respects the level of the wefts. This functor  $Weft(\Phi)$  is defined recursively on the level:

At level 0,  $Weft(\Phi)$  is just  $\Phi$ .

Let  $n \geq 1$ , and assume that  $Weft(\Phi)$  is defined on the category of  $\mathcal{A}$ -wefts of level  $\leq n-1$ . For all constraint  $\Gamma = (\chi, (\gamma, \delta))$  of level  $\leq n-1$  over a point  $A_0$  of  $\mathcal{A}$ , let  $Weft(\Phi)(\Gamma)$  denote the constraint of level  $\leq n-1$  on  $\Phi(A)$  with body  $\Phi(\chi)$  and potential  $(Weft(\Phi)(\gamma), Weft(\Phi)(\delta))$ . Then the image by  $Weft(\Phi)$  of an  $\mathcal{A}$ -weft  $\mathbf{S}$  of level  $\leq n$  is the  $\mathcal{A}'$ -weft with support  $\Phi(\underline{\mathbf{S}})$  and constraints the  $Weft(\Phi)(\Gamma)$  for all the constraints  $\Gamma$  of  $\mathbf{S}$ .

#### 3 Projective sketches

 $\mathcal{A}$ -wefts were defined in 2 for any category  $\mathcal{A}$ , as well as the realizations which they specify. In our examples, the category  $\mathcal{A}$  is the category  $\mathcal{A}mbi$  of ambigraphs. However we will see in [guide2] and [guide3] that other categories may prove useful.

Let us consider once more the category  $\mathcal{A}mbi$ . According to its definition, an ambigraph  $\mathcal{G}$  is made up of sets  $\mathcal{G}(\mathsf{Pt})$ ,  $\mathcal{G}(\mathsf{Ar})$ , etc. and maps  $\mathcal{G}(\mathsf{dom}): \mathcal{G}(\mathsf{Ar}) \to \mathcal{G}(\mathsf{Pt})$ ,  $\mathcal{G}(\mathsf{codom}): \mathcal{G}(\mathsf{Ar}) \to \mathcal{G}(\mathsf{Pt})$ , etc. which satisfy some properties. Actually, as is suggested by our notation,  $\mathcal{G}$  can be identified with a set-valued realization of a weft  $\mathbf{E}_{\mathcal{A}mbi}$ . It is a  $\mathit{Comp}$ -weft (i.e. a weft of compositive graphs; the definition of the category  $\mathit{Comp}$  is given in A.3). The support of  $\mathbf{E}_{\mathcal{A}mbi}$  is made up of points Pt, Ar, etc. and arrows dom: Ar  $\to$  Pt, codom: Ar  $\to$  Pt, etc. In addition, the weft of compositive graphs  $\mathbf{E}_{\mathcal{A}mbi}$  is a projective sketch, because all its constraints are constraints of projective limits (see A.5).

Sketches were introduced by Ehresmann [Ehresmann 66, Ehresmann 67a, Ehresmann 67b, Ehresmann 68]. In [Coppey & Lair 84, Coppey & Lair 88] they are presented in an elementary way. Here we focus on projective sketches.

The set-valued realizations of a projective sketch  $\mathbf{E}$  are called the *models* of  $\mathbf{E}$ . We will see that it is easy to define the *category*  $\mathcal{M}od(\mathbf{E})$  of models of  $\mathbf{E}$ . The category  $\mathcal{M}od(\mathbf{E}_{\mathcal{A}mbi})$  is equivalent to the category  $\mathcal{A}mbi$  of ambigraphs. We say that the projective sketch  $\mathbf{E}_{\mathcal{A}mbi}$  sketches the category  $\mathcal{A}mbi$ .

A category  $\mathcal{A}$  is *projectively sketchable* if it is equivalent to the category of models of some projective sketch. Since we will be interested in  $\mathcal{A}$ -wefts for projectively sketchable categories  $\mathcal{A}$ , we now study projective sketches and their categories of models.

In 3.1 and 3.4, we build the projective sketch  $\mathbf{E}_{\mathcal{A}mbi}$ . In 3.2 projective sketches are defined. Their models, with the corresponding homomorphisms, are defined in 3.3. A table in 3.5 sums up our terminology about wefts and sketches.

#### 3.1 A compositive graph for ambigraphs

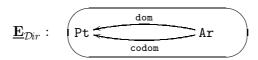
Ambigraphs are defined in 2.1 from compositive graphs, which themselves are defined in A.2 from directed graphs, which are defined in A.1. Hence we now build three projective sketches, each one extending the previous one: first  $\mathbf{E}_{\mathcal{D}ir}$  for directed graphs, then  $\mathbf{E}_{\mathcal{C}omp}$  for compositive graphs, and finally  $\mathbf{E}_{\mathcal{A}mbi}$  for ambigraphs. Their supports, which are compositive graphs, are described in this section (these descriptions are abbreviated, as in 2.1). Their constraints will be described in section 3.4.

To begin with, for directed graphs, let  $\underline{\mathbf{E}}_{\mathcal{D}ir}$  be the following (simple) compositive graph:

Compositive graph  $\mathbf{E}_{\mathcal{D}ir}$ :

points: Pt, Ar,

*arrows*:  $dom : Ar \rightarrow Pt$ ,  $codom : Ar \rightarrow Pt$ .



Each directed graph  $\mathcal{G}$  determines a functor from  $\underline{\mathbf{E}}_{\mathcal{D}ir}$  towards  $\mathcal{S}et$ :

*points*:  $Pt \mapsto \mathcal{G}(Pt)$ ,  $Ar \mapsto \mathcal{G}(Ar)$ ,

 $arrows: dom \mapsto \mathcal{G}(dom), codom \mapsto \mathcal{G}(codom).$ 

For compositive graphs, let  $\underline{\mathbf{E}}_{\mathcal{C}omp}$  be the following compositive graph:

Compositive graph  $\mathbf{E}_{\mathcal{C}omp}$ :

extends:  $\underline{\mathbf{E}}_{Dir}$ ,

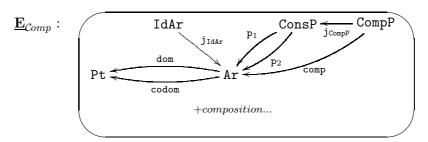
points: IdAr, ConsP, CompP,

arrows: comp: CompP  $\rightarrow$  Ar, p<sub>1</sub>: ConsP  $\rightarrow$  Ar, p<sub>2</sub>: ConsP  $\rightarrow$  Ar,

 $\mathtt{j}_{\mathtt{IdAr}} : \mathtt{IdAr} \to \mathtt{Ar}, \ \mathtt{j}_{\mathtt{CompP}} : \mathtt{CompP} \to \mathtt{ConsP},$ 

 $\mathit{composition} \colon \quad \mathtt{dom} \circ \mathtt{j}_{\mathtt{IdAr}} \! = \! \mathtt{codom} \circ \mathtt{j}_{\mathtt{IdAr}}, \, \mathtt{dom} \circ \mathtt{p}_2 \! = \! \mathtt{codom} \circ \mathtt{p}_1,$ 

 $\mathtt{dom} \circ \mathtt{comp} = \mathtt{dom} \circ (\mathtt{p}_1 \circ \mathtt{j}_{\mathtt{CompP}}), \ \mathtt{codom} \circ \mathtt{comp} = \mathtt{codom} \circ (\mathtt{p}_2 \circ \mathtt{j}_{\mathtt{CompP}}).$ 



Similarly, each compositive graph  $\mathcal{G}$  determines a functor from  $\underline{\mathbf{E}}_{Comp}$  towards Set. Composition in  $\underline{\mathbf{E}}_{Comp}$  corresponds to the following properties of  $\mathcal{G}$ :

- $g: G \to G$  for all identity arrow g of  $\mathcal{G}$ ;
- the codomain of  $g_1$  is equal to the domain of  $g_2$  for all consecutive pair  $(g_1, g_2)$  of  $\mathcal{G}$ ;
- the domain of  $g_2 \circ g_1$  is the domain of  $g_1$ , and its codomain is the codomain of  $g_2$  for all composable pair  $(g_1, g_2)$  of  $\mathcal{G}$ .

However, there is *nothing* in  $\underline{\mathbf{E}}_{\mathcal{C}omp}$  which corresponds to the following properties:

- the consecutive pairs are all the pairs  $(g_1, g_2)$  such that the codomain of  $g_1$  is equal to the domain of  $g_2$ ;
- $\bullet \ \ \text{the maps} \ \mathcal{G}(\mathtt{j}_{\mathtt{IdAr}}): \mathcal{G}(\mathtt{IdAr}) \to \mathcal{G}(\mathtt{Ar}) \ \ \text{and} \ \ \mathcal{G}(\mathtt{j}_{\mathtt{CompP}}): \mathcal{G}(\mathtt{CompP}) \to \mathcal{G}(\mathtt{ConsP}) \ \ \text{are} \ \ \mathit{injections}.$

We will see in 3.4 how constraints can be added to  $\underline{\mathbf{E}}_{\mathcal{C}omp}$  in order to lay down these properties.

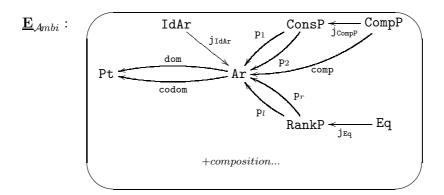
Finally, for ambigraphs, let  $\underline{\mathbf{E}}_{Ambi}$  be the following compositive graph:

Compositive graph  $\underline{\mathbf{E}}_{Ambi}$ :

extends:  $\underline{\mathbf{E}}_{\mathcal{C}omp}$ , points: RankP, Eq,

arrows:  $p_l: RankP \rightarrow Ar, p_r: RankP \rightarrow Ar, j_{Eq}: Eq \rightarrow RankP,$ 

composition:  $dom \circ p_l = dom \circ p_r$ ,  $codom \circ p_l = codom \circ p_r$ 



In a similar way, each ambigraph  $\mathcal{G}$  determines a functor from  $\underline{\mathbf{E}}_{\mathcal{A}mbi}$  towards  $\mathcal{S}et$ . We will see in 3.4 how constraints can be added to  $\underline{\mathbf{E}}_{\mathcal{A}mbi}$  in order to lay down the following properties of  $\mathcal{G}$ :

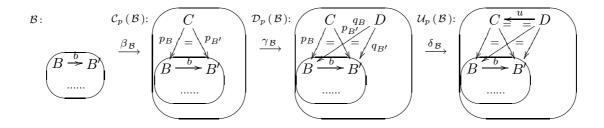
- the consecutive pairs are all the pairs  $(g_1, g_2)$  such that the codomain of  $g_1$  is equal to the domain of  $g_2$ ;
- the pairs with the same rank are all the pairs  $(g_l, g_r)$  such that the rank of  $g_r$  is equal to the rank of  $g_l$ ;
- the maps  $\mathcal{G}(j_{\mathtt{IdAr}})$ ,  $\mathcal{G}(j_{\mathtt{CompP}})$  and  $\mathcal{G}(j_{\mathtt{Eq}})$  are injections.

#### 3.2 Projective cones and projective sketches

The constraints which we will add to  $\underline{\mathbf{E}}_{Comp}$  and  $\underline{\mathbf{E}}_{Ambi}$  have a special shape, they are called distinguished projective cones, and they lay down properties of projective limits. Definitions of a typical projective cone, a projective cone and a limit projective cone can be found in A.5.

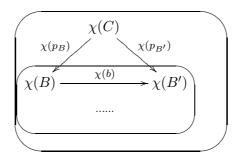
**Definition 5** Let  $\mathcal{B}$  be a compositive graph. A  $\mathcal{B}$ -distinguished projective cone is a constraint which has level 0 and the following potential:

- the compositive graph C is the typical  $\mathcal{B}$ -projective cone  $C_p(\mathcal{B})$ ;
- the compositive graph  $\mathcal{D} = \mathcal{D}_p(\mathcal{B})$  extends  $\mathcal{C}_p(\mathcal{B})$  by a second typical  $\mathcal{B}$ -projective cone, i.e. by: a point D, an arrow  $q_B: D \to B$  for all point B of  $\mathcal{B}$ , and composition  $b \circ q_B = q_{B'}$  for all arrow  $b: B \to B'$  of  $\mathcal{B}$ ;
- the compositive graph  $\mathcal{U} = \mathcal{U}_p(\mathcal{B})$  extends  $\mathcal{D}_p(\mathcal{B})$  by an arrow  $u : D \to C$  and composition  $p_B \circ u = q_B$  for all point B of  $\mathcal{B}$ ;
- the functors  $\gamma_{\mathcal{B}}: \mathcal{C}_p(\mathcal{B}) \to \mathcal{D}_p(\mathcal{B})$  and  $\delta_{\mathcal{B}}: \mathcal{D}_p(\mathcal{B}) \to \mathcal{U}_p(\mathcal{B})$  are the extensions.



Let  $\mathcal{G}$  be a compositive graph. For any given compositive graph  $\mathcal{B}$ , a distinguished  $\mathcal{B}$ -projective cone in  $\mathcal{G}$  is characterized by its body  $\chi: \mathcal{C}_p(\mathcal{B}) \to \mathcal{G}$ . Often it is this body which is called the distinguished  $\mathcal{B}$ -projective cone, rather than the complete constraint.

A distinguished  $\mathcal{B}$ -projective cone  $\chi: \mathcal{C}_p(\mathcal{B}) \to \mathcal{G}$  is outlined as follows:



**Definition 6** A projective sketch  $\mathbf{E}$  is a compositive graph  $\underline{\mathbf{E}}$ , called the support of  $\mathbf{E}$ , together with a set of distinguished projective cones.

The points, arrows,... of a projective sketch are the points, arrows,... of its support.

A homomorphism of projective sketches  $\rho : \mathbf{E} \to \mathbf{E}'$  is a functor of compositive graphs  $\underline{\rho} : \underline{\mathbf{E}} \to \underline{\mathbf{E}}'$  such that for all distinguished projective cone  $\chi : \mathcal{C} \to \underline{\mathbf{E}}$  of  $\mathbf{E}$ , the composed arrow  $\underline{\rho} \circ \chi : \mathcal{C} \to \underline{\mathbf{E}}'$  is a distinguished projective cone of  $\mathbf{E}'$ .

Equivalently, projective sketches are the wefts of compositive graphs with level  $\leq 1$  and constraints which are distinguished projective cones. And homomorphisms of projective sketches are homomorphisms of Comp-wefts between projective sketches.

This defines the category of projective sketches (for size questions, see [ref1] and [ref2]):

 $\mathcal{P}sk$ .

#### 3.3 Models of a projective sketch

Let **E** be a projective sketch. The definition of models of **E** is given below: they are the realizations of the *Comp*-weft **E**. But here, in addition, we are able to define, in a simple way, homomorphisms between these models. Since we will only need set-valued models, we do not consider models of **E** towards other compositive graphs, though it would not be any more difficult.

**Definition 7** A model of  $\mathbf{E}$  is a functor from the compositive graph  $\underline{\mathbf{E}}$  towards  $\mathcal{S}et$  such that the image of each distinguished projective cone of  $\mathbf{E}$  is a limit projective cone.

A homomorphism of models of  ${\bf E}$  is a natural transformation (see A.4) between the sublying functors .

The definition of a homomorphism  $h: \mu_1 \to \mu_2$  of models of  $\mathbf{E}$  does not involve the distinguished projective cones of  $\mathbf{E}$ : it is a family of maps  $h(E): \mu_1(E) \to \mu_2(E)$  for all point E of  $\mathbf{E}$ , such that  $\mu_2(e) \circ h(E_1) = h(E_2) \circ \mu_1(e)$  for all arrow  $e: E_1 \to E_2$  of  $\mathbf{E}$ . However such a homomorphism behaves well with respect to distinguished projective cones of  $\mathbf{E}$ : indeed, the property of projective limits ensures that, given a distinguished projective cone  $\chi: \mathcal{C}_p(\mathcal{B}) \to \underline{\mathbf{E}}$  of  $\mathbf{E}$ , the restriction of h to the image of  $\mathcal{B}$  determines h upon the image of the whole  $\mathcal{C}_p(\mathcal{B})$ .

The models of  $\mathbf{E}$  are the points, and the homomorphisms between these models are the arrows, of the category:

 $\mathcal{M}od(\mathbf{E})$ .

Let  $\rho: \mathbf{E} \to \mathbf{E}'$  be a homomorphism of projective sketches. Then:

$$\mathcal{M}od(\rho): \mathcal{M}od(\mathbf{E}') \to \mathcal{M}od(\mathbf{E})$$

denotes the functor associated with  $\rho$ . It maps each model  $\mu' : \mathbf{E}' \to \mathcal{S}et$  to the model  $\mu' \circ \rho : \mathbf{E} \to \mathcal{S}et$ . In this way we get a contravariant functor:

$$\mathcal{M}od(-): \mathcal{P}sk \longrightarrow \mathcal{C}at$$
.

A model  $\mu$  of **E** is denoted:

$$\mu: \mathbf{E} \longrightarrow \mathcal{S}et$$
.

In a similar way, a weft **S** of models of **E**, i.e. a  $\mathcal{M}od(\mathbf{E})$ -weft, is denoted:

$$\mathbf{S}: \mathbf{E} \xrightarrow{\frown} \mathcal{S}et$$
.

**Definition 8** Let E be a point of  $\mathbf{E}$ .

Let  $\mu : \mathbf{E} \longrightarrow \mathcal{S}et$  be a model of  $\mathbf{E}$ . An *ingredient* of  $\mu$  of *nature* E is an element of the set  $\mu(E)$ . Let  $\mathbf{S} : \mathbf{E} \xrightarrow{\frown} \mathcal{S}et$  be a  $\mathcal{M}od(\mathbf{E})$ -weft. An *ingredient* of  $\mathbf{S}$  of *nature* E is an ingredient of nature E of its support  $\underline{\mathbf{S}}$ .

Some categories can be described as the category of models of a projective sketch:

**Definition 9** Let  $\mathcal{A}$  be a category. It is *projectively sketchable* if it is equivalent to the category of models of some projective sketch  $\mathbf{E}$ , *i.e.* if:

$$\mathcal{A} \simeq \mathcal{M}od(\mathbf{E})$$
.

Then we say that **E** sketches A.

#### 3.4 A projective sketch for ambigraphs

Let us now add to the compositive graph  $\underline{\mathbf{E}}_{\mathcal{A}mbi}$  some distinguished projective cones in order to get a projective sketch  $\mathbf{E}_{\mathcal{A}mbi}$  which sketches ambigraphs, *i.e.* such that the category of models of  $\mathbf{E}_{\mathcal{A}mbi}$  is equivalent to  $\mathcal{A}mbi$ . As in 3.1, let us first look at directed graphs and compositive graphs.

The compositive graph  $\underline{\mathbf{E}}_{\mathcal{D}ir}$  is the support of a projective sketch without any distinguished projective cone:

Projective sketch  $\mathbf{E}_{\mathcal{D}ir}$ :

support:  $\underline{\mathbf{E}}_{\mathcal{D}ir}$ .

**Proposition 1** The category of models of the projective sketch  $\mathbf{E}_{Dir}$  is isomorphic to the category Dir of directed graphs:

$$\mathcal{M}od(\mathbf{E}_{\mathcal{D}ir}) \cong \mathcal{D}ir$$
.

About the proof. We have already noticed that each directed graph  $\mathcal{G}$  determines a functor  $\Phi_{Dir}(\mathcal{G})$  from  $\underline{\mathbf{E}}_{Dir}$  towards Set, such that:

•  $\Phi_{Dir}(\mathcal{G})(E) = \mathcal{G}(E)$  for all point E of  $\underline{\mathbf{E}}_{Dir}$  (i.e.  $E = \mathsf{Pt}$  or  $\mathsf{Ar}$ );

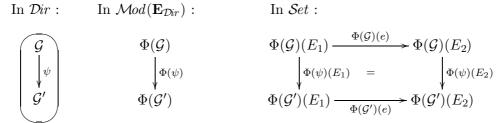
•  $\Phi_{Dir}(\mathcal{G})(e) = \mathcal{G}(e)$  for all arrow e of  $\underline{\mathbf{E}}_{Dir}$  (i.e. e = dom or codom).

Similarly, each homomorphism of directed graphs  $\psi: \mathcal{G} \to \mathcal{G}'$  determines a family of maps:

• 
$$\Phi_{Dir}(\psi)(E) = \psi(E) : \mathcal{G}(E) \to \mathcal{G}'(E)$$
 for all point  $E$  of  $\underline{\mathbf{E}}_{Dir}$ .

Let us check that these maps build up a homomorphism  $\Phi_{\mathcal{D}ir}(\psi):\Phi_{\mathcal{D}ir}(\mathcal{G})\to\Phi_{\mathcal{D}ir}(\mathcal{G}')$  of models of  $\mathbf{E}_{\mathcal{D}ir}$ , i.e. that for all arrow  $e:E_1\to E_2$  of  $\mathbf{E}_{\mathcal{D}ir}$ :

$$\Phi_{\mathcal{D}ir}(\psi)(E_2) \circ \Phi_{\mathcal{D}ir}(\mathcal{G})(e) = \Phi_{\mathcal{D}ir}(\mathcal{G}')(e) \circ \Phi_{\mathcal{D}ir}(\psi)(E_1) : \Phi_{\mathcal{D}ir}(\mathcal{G})(E_1) \to \Phi_{\mathcal{D}ir}(\mathcal{G}')(E_2)$$
.



Indeed, from the definition of  $\Phi_{Dir}$ , it means that for all arrow  $e: E_1 \to E_2$  of  $\mathbf{E}_{Dir}$ :

$$\psi(E_2) \circ \mathcal{G}(e) = \mathcal{G}'(e) \circ \psi(E_1) : \mathcal{G}(E_1) \to \mathcal{G}'(E_2)$$
.

Since  $\mathbf{E}_{\mathcal{D}ir}$  has only two arrows dom and codom, both with rank  $Ar \to Pt$ , the required property is the following one:

$$\psi(\mathsf{Pt}) \circ \mathcal{G}(\mathsf{dom}) = \mathcal{G}'(\mathsf{dom}) \circ \psi(\mathsf{Ar}) \text{ and } \psi(\mathsf{Pt}) \circ \mathcal{G}(\mathsf{codom}) = \mathcal{G}'(\mathsf{codom}) \circ \psi(\mathsf{Ar}).$$

It means that for all arrow  $g: G_1 \to G_2$  of  $\mathcal{G}$ , the arrow  $\psi(\operatorname{Ar})(g)$  of  $\mathcal{G}'$  has domain  $\psi(\operatorname{Pt})(G_1)$  and codomain  $\psi(\operatorname{Pt})(G_2)$ , which is indeed true, since  $\psi$  is a homomorphism of directed graphs.

In this way we get a functor  $\Phi_{Dir}: Dir \to \mathcal{M}od(\mathbf{E}_{Dir})$ . Now it is easy to check that  $\Phi_{Dir}$  is an isomorphism.

 $\Diamond$ 

Thanks to this result, we may identify directed graphs and models of  $\mathbf{E}_{\mathcal{D}ir}$ . For all directed graph  $\mathcal{G}$ , we use  $\mathcal{G}$  for  $\Phi_{\mathcal{D}ir}(\mathcal{G})$ . In the same way, for all homomorphism of directed graphs  $\psi: \mathcal{G} \to \mathcal{G}'$ , we use  $\psi$  for  $\Phi_{\mathcal{D}ir}(\psi)$ .

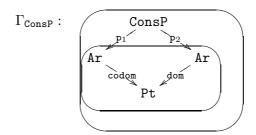
Now let us add to the compositive graph  $\underline{\mathbf{E}}_{Comp}$  some constraints, in order to lay down the following properties of each compositive graph  $\mathcal{G}$ :

- the consecutive pairs are all the pairs  $(g_1, g_2)$  of arrows of  $\mathcal{G}$ , such that the codomain of  $g_1$  is equal to the domain of  $g_2$ ;
- the maps  $\mathcal{G}(j_{IdAr})$  and  $\mathcal{G}(j_{CompP})$  are injections.

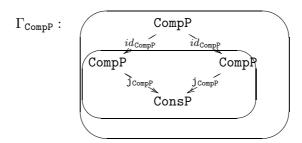
These properties can be stated as projective limits in the category of sets. More precisely, the first one can be stated as a *pullback* and the second one as a *monomorphism*.

This is why we add to  $\underline{\mathbf{E}}_{Comp}$  three distinguished projective cones:

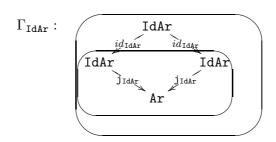
• the point ConsP is the vertex of the distinguished projective cone  $\Gamma_{\text{ConsP}}$ :



• the point CompP is the vertex of the distinguished projective cone  $\Gamma_{\text{CompP}}$ :



• the point IdAr is the vertex of the distinguished projective cone  $\Gamma_{IdAr}$ :



Hence, with "d.p.c." for "distinguished projective cones":

PROJECTIVE SKETCH  $\mathbf{E}_{\mathcal{C}omp}$ :

support:  $\underline{\mathbf{E}}_{\mathcal{C}omp}$ ,

d.p.c.:  $\Gamma_{\texttt{ConsP}}$ ,  $\Gamma_{\texttt{CompP}}$ ,  $\Gamma_{\texttt{IdAr}}$ .

This is an abbreviated notation: indeed, the identity arrows  $id_{\texttt{CompP}}$  and  $id_{\texttt{IdAr}}$ , which occur in the distinguished projective cones  $\Gamma_{\texttt{CompP}}$  and  $\Gamma_{\texttt{IdAr}}$ , should be added to the support of  $\mathbf{E}_{\textit{Comp}}$ . Moreover, it should be noted, since it is not clear from our notations, that a point in a projective sketch can be the vertex of several distinguished projective cones.

**Proposition 2** The category of models of the projective sketch  $\mathbf{E}_{Comp}$  is equivalent to the category Comp of compositive graphs:

$$\mathcal{M}od(\mathbf{E}_{\mathcal{C}omn}) \simeq \mathcal{C}omp$$
.

About the proof. A functor  $\Phi_{Comp}: Comp \to \mathcal{M}od(\mathbf{E}_{Comp})$  is easily defined using the functor  $\Phi_{\mathcal{D}ir}: \mathcal{D}ir \to \mathcal{M}od(\mathbf{E}_{\mathcal{D}ir})$  from the proof of proposition 1. Then it is easy to check that  $\Phi_{Comp}$  is an equivalence.

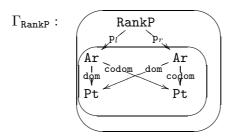
The functor  $\Phi_{\mathcal{D}ir}$  is an isomorphism, however the functor  $\Phi_{\mathcal{C}omp}$  is only an equivalence. This comes from the fact that  $\mathbf{E}_{\mathcal{C}omp}$ , unlike  $\mathbf{E}_{\mathcal{D}ir}$ , has distinguished projective cones. The interpretation of such a cone by a model of  $\mathbf{E}_{\mathcal{C}omp}$  is any limit projective cone. However, we will identify each compositive graph with the corresponding model of  $\mathbf{E}_{\mathcal{C}omp}$ .

Now let us add to the compositive graph  $\underline{\mathbf{E}}_{\mathcal{A}mbi}$  some constraints corresponding to the following properties of each ambigraph  $\mathcal{G}$ :

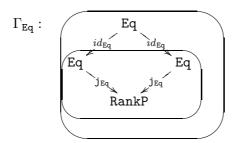
- $\mathcal{G}(ConsP)$  (resp.  $\mathcal{G}(RankP)$ ) is the set of all consecutive pairs (resp. pairs with the same rank) of  $\mathcal{G}$ ;
- the maps  $\mathcal{G}(j_{IdAr})$ ,  $\mathcal{G}(j_{CompP})$  and  $\mathcal{G}(j_{Eq})$  are injections.

These properties can be stated as projective limits in the category of sets. So that we now add to  $\underline{\mathbf{E}}_{Ambi}$  five distinguished projective cones:

- the point ConsP is the vertex of the distinguished projective cone  $\Gamma_{\text{ConsP}}$  as above;
- the point RankP is the vertex of the distinguished projective cone  $\Gamma_{RankP}$ :



- the point IdAr (resp. CompP) is the vertex of the distinguished projective cone  $\Gamma_{\text{IdAr}}$  (resp.  $\Gamma_{\text{CompP}}$ ) as above;
- the point Eq is the vertex of the distinguished projective cone  $\Gamma_{Eq}$ :



Hence:

Projective sketch  $\mathbf{E}_{\mathcal{A}mbi}$ :

support:  $\underline{\mathbf{E}}_{\mathcal{A}mbi}$ ,

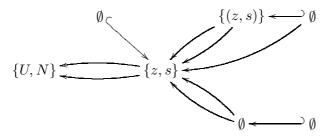
d.p.c.:  $\Gamma_{\text{ConsP}}$ ,  $\Gamma_{\text{RankP}}$ ,  $\Gamma_{\text{IdAr}}$ ,  $\Gamma_{\text{CompP}}$ ,  $\Gamma_{\text{Eq}}$ .

**Proposition 3** The category of models of the projective sketch  $\mathbf{E}_{Ambi}$  is equivalent to the category Ambi of ambigraphs:

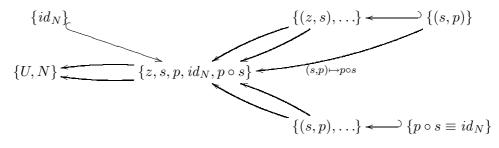
$$\mathcal{M}od(\mathbf{E}_{\mathcal{A}mbi}) \simeq \mathcal{A}mbi$$
.

About the proof. Similar to the proof of proposition 2.  $\Diamond$ 

**Example 11** The ambigraph  $\mathcal{G}_s$  in example 1 corresponds to a model of  $\mathbf{E}_{Ambi}$ :



In the same way, the ambigraph  $\mathcal{G}_{s,p}$  in example 1 corresponds to a model of  $\mathbf{E}_{\mathcal{A}mbi}$ :



#### 3.5 About wefts and sketches

Projective sketches are a special kind of wefts (*Comp*-wefts with limit constraints), and the models of a projective sketch are its set-valued realizations.

However, projective sketches will play a very specific role among wefts, since they will *sketch* wefts. More precisely, we will consider  $\mathcal{A}$ -wefts for categories  $\mathcal{A}$  which are projectively sketchable, i.e. such that:

$$\mathcal{A}\simeq\mathcal{M}\!\mathit{od}(\mathbf{E})$$

for some projective sketch  $\mathbf{E}$ . This is the reason why we have introduced, for dealing with wefts, a terminology which is different from the usual one for sketches. The table below sums up these differences.

SKETCHES	WEFTS		
	(A  is a category)		
	and $A$ is a point of $\mathcal{A}$ )		
projective sketch	$\mathcal{A} ext{-weft}$		
${f E}$	$\mathbf{S}$		
$\bmod el\ \mathbf{E}$	realization of <b>S</b> towards $A$		
$\mu: \mathbf{E}  o \mathcal{S}et$	$\omega:\mathbf{S} o A$		
the category	the <b>set</b>		
$\mathcal{M}\!od(\mathbf{E})$	$\operatorname{Real}(\mathbf{S},A)$		

#### 4 Blowing up projective sketches

According to 2, the  $\mathcal{A}$ -wefts (for any category  $\mathcal{A}$ ) can be used for specifying. Quite often,  $\mathcal{A}$  is the category  $\mathcal{A}mbi$  of ambigraphs. According to 3, the category  $\mathcal{A}mbi$  is projectively sketchable: it is equivalent to the category  $\mathcal{M}od(\mathbf{E}_{\mathcal{A}mbi})$  of models of the projective sketch  $\mathbf{E}_{\mathcal{A}mbi}$ .

In [guide2] we will consider  $\mathcal{A}$ -wefts for other categories  $\mathcal{A}$ , which will also be projectively sketchable, *i.e.* equivalent to  $\mathcal{M}od(\mathbf{F})$  for some projective sketch  $\mathbf{F}$ . We will focus on the special case where there is a homomorphism  $\rho: \mathbf{F} \to \mathbf{E}_{\mathcal{A}mbi}$ .

The simplest among these projective sketches are the *blow-ups*  $\mathbf{E}_{\mathcal{A}mbi} \setminus \mathcal{I}$  of  $\mathbf{E}_{\mathcal{A}mbi}$  by an ambigraph  $\mathcal{I}$ , with the homomorphism called *fibration*  $\mathbf{E}_{\mathcal{A}mbi} \setminus \mathcal{I} : \mathbf{E}_{\mathcal{A}mbi} \setminus \mathcal{I} \to \mathbf{E}_{\mathcal{A}mbi}$ . Moreover, the blow-up  $\mathbf{E}_{\mathcal{A}mbi} \setminus \mathcal{I}$  sketches the category of  $\mathcal{I}$ -indexations, where an  $\mathcal{I}$ -indexation is an ambifunctor with codomain  $\mathcal{I}$ .

The blow-up of a projective sketch by one of its models is studied by Lair in [Lair 71, Lair 77]; this notion is similar, for sketches, to the well-known notion of discrete fibration associated to a functor from a category towards Set, or to the compositive graph of hypermorphisms introduced by Ehresmann in [Ehresmann 65]. The aim of this section is to define and study the blow-ups of any projective sketch. Let:

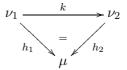
**E** be a projective sketch and  $\mu$  a model of **E**.

In 4.1 we define the category  $\mathcal{M}od(\mathbf{E})/\mu$  of  $\mu$ -indexations of models of  $\mathbf{E}$ . Then in 4.2 we define the projective sketch  $\mathbf{E}\backslash\mu$  blow-up of  $\mathbf{E}$  by  $\mu$  and we prove its main property: both categories  $\mathcal{M}od(\mathbf{E})/\mu$  and  $\mathcal{M}od(\mathbf{E}\backslash\mu)$  are equivalent. In 4.3 we generalize this result to  $\mu$ -indexations of  $\mathcal{M}od(\mathbf{E})$ -wefts, and we introduce the loose homomorphisms of wefts. In 4.4 we consider an example: the blow-up of  $\mathbf{E}_{\mathcal{A}mbi}$  by each model  $\mathcal{Y}_{\mathbf{E}_{\mathcal{A}mbi}}(E)$  given by its Yoneda counter-model  $\mathcal{Y}_{\mathbf{E}_{\mathcal{A}mbi}}: \mathbf{E}_{\mathcal{A}mbi} \longrightarrow \mathcal{M}od(\mathbf{E}_{\mathcal{A}mbi})$ .

#### 4.1 Indexations

**Definition 10** An indexation by  $\mu$  (or  $\mu$ -indexation) of a model  $\nu$  of **E** is a homomorphism  $h: \nu \to \mu$  of models of **E**.

Let  $h_1: \nu_1 \to \mu$  and  $h_2: \nu_2 \to \mu$  be two  $\mu$ -indexations. A  $\mu$ -triangle from  $h_1$  towards  $h_2$  is a homomorphism  $k: \nu_1 \to \nu_2$  of models of  $\mathbf E$  such that  $h_2 \circ k = h_1$ .



It is easy to define the category:

$$\mathcal{M}od(\mathbf{E})/\mu$$
.

Its points are the  $\mu$ -indexations in  $\mathcal{M}od(\mathbf{E})$  and its arrows are the  $\mu$ -triangles.

Let  $h: \nu \to \mu$  be an indexation of  $\nu$  by  $\mu$  in the category  $\mathcal{M}od(\mathbf{E})$ . Let x and y be two ingredients of  $\mu$  and  $\nu$  respectively, both with the same nature E. If h(E)(y) = x we say that x is the index of y.

#### Example 12 Let us consider:

Projective sketch  $\mathbf{E}_{\mathcal{M}ap}$ :

points: E, E',arrow:  $e: E \rightarrow E'.$ 

Each map  $f: X \to X'$  determines a model  $\mathcal{I}_f$  of  $\mathbf{E}_{\mathcal{M}ap}$ , and vice-versa:

MODEL  $\mathcal{I}_f : \mathbf{E}_{\mathcal{M}ap} \to \mathcal{S}et:$ points:  $E \mapsto X, E' \mapsto X',$ arrow:  $e \mapsto f.$ 

An indexation by  $\mathcal{I}_f$  of a model  $\mathcal{I}_g$  of  $\mathbf{E}_{\mathcal{M}ap}$  (where  $g:Y\to Y'$  is some map) is a homomorphism  $h:\mathcal{I}_g\to\mathcal{I}_f$  i.e. a pair of maps  $(h:Y\to X\ ,\ h':Y'\to X')$  such that  $f\circ h=h'\circ g$ :

$$Y \xrightarrow{h} X$$

$$g \downarrow \qquad = \qquad \downarrow f$$

$$Y' \xrightarrow{h'} X'$$

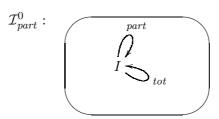
**Example 13** Let us consider the following category  $\mathcal{F}unc$ : its points are sets and its arrows are *functions* (where a function, unlike a map, can be partial). In this example, we consider the homomorphisms from a directed graph  $\mathcal{G}$  towards the category  $\mathcal{F}unc$ .

Moreover, we wish to say, for all arrow g of  $\mathcal{G}$ , whether the interpretation of g should be total or partial (here "partial" means "strictly partial", *i.e.* not total). For this purpose, we map each arrow of  $\mathcal{G}$  to the symbol tot if we wish its interpretation to be total, or to the symbol part if we wish it to be partial. More precisely, we build a  $\mathcal{I}_{part}^0$ -indexation, where  $\mathcal{I}_{part}^0$  is the following directed graph:

DIRECTED GRAPH  $\mathcal{I}^0_{part}$ :

point: I,

 $\mathit{arrows}\colon \ \mathit{part}:I\to I,\ \mathit{tot}:I\to I.$ 



The  $\mathcal{I}^0_{part}$ -indexation of the directed graph sublying to  $\mathcal{F}unc$  is the following:

Homomorphism  $h_{part}^0: \mathcal{F}unc \to \mathcal{I}_{part}^0$ :

points:  $X \mapsto I$  for all set X

arrows:  $f \mapsto tot$  for all total function f,  $f \mapsto part$  for all partial function f.

Let us now consider the directed graph  $\mathcal{G}_{s,p}^0$  and its  $\mathcal{I}_{part}^0$ -indexation  $h_{s,p}^0$ :

DIRECTED GRAPH  $\mathcal{G}_{s,p}^0$ :

points: U, N,

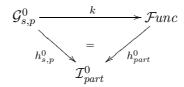
arrows:  $z: U \to N, s: N \to N, p: N \to N.$ 

Homomorphism  $h_{s,p}^0:\mathcal{G}_{s,p}^0\to\mathcal{I}_{part}^0$ :

points:  $U \mapsto I, \ \stackrel{\frown}{N} \mapsto \stackrel{\frown}{I},$ 

arrows:  $z \mapsto tot, s \mapsto tot, p \mapsto part.$ 

Let k be any homomorphism of directed graphs from  $\mathcal{G}_{s,p}^0$  towards the category  $\mathcal{F}unc$ ; then the interpretation of the arrows z, s and p by k are arbitrary functions. Now, more precisely, let k be a  $\mathcal{I}_{part}^0$ -triangle from  $h_{s,p}^0$  towards  $h_{part}^0$ ; then the interpretation of the arrows z and s by k are total functions whereas the interpretation of p is a partial function.



**Example 14** Let us try to go further, by adding equations to the example above. For this purpose, the directed graph  $\mathcal{I}_{part}^0$  can be extended by the following ambigraph  $\mathcal{I}_{part}$ :

Ambigraph  $\mathcal{I}_{part}$ :

extends:  $\mathcal{I}_{part}^{0},$  identity arrow:  $tot = id_{I},$ 

 $composed\ arrows:\ \ tot\circ tot=tot,\ part\circ part=part\circ tot=tot\circ part=part,$ 

equations:  $tot \equiv tot, part \equiv part.$ 

The identity arrow of  $\mathcal{I}_{part}$  means that identity functions are total.

The composed arrows of  $\mathcal{I}_{part}$  mean that, given two functions f and g, the composed function  $g \circ f$  is total when f and g are total, and otherwise  $g \circ f$  is generally partial.

The equations of  $\mathcal{I}_{part}$  mean that for two functions to be equal, they must be either both total or both partial.

It should be noticed that here composition and equations play fairly different roles, unlike in previous examples: the role of the equation  $p \circ s \equiv id_N$  in the ambigraph  $\mathcal{G}_{s,p}$  from example 1 is similar to the role of the composition  $p \circ s = id_N$  in the compositive graph  $\mathcal{G}'_{s,p}$  from example 2; indeed, both are interpreted as equalities in categories. On the contrary, the ambigraph  $\mathcal{I}_{part}$  is not used as the domain of an ambifunctor towards a category; it indexes other ambigraphs  $\mathcal{G}$ , and each of its ingredients gives some information about the required interpretation of some ingredients of  $\mathcal{G}$ . We will see in [guide2] and [guide3] more sophisticated examples, where equations will be used to express more subtle notions of "equality".

Here it is impossible to extend the indexation of directed graphs  $h_{part}^0: \mathcal{F}unc \to \mathcal{I}_{part}^0$  in an indexation of ambigraphs  $h_{part}: \mathcal{F}unc \to \mathcal{I}_{part}$ . Indeed, when at least one function in a pair of consecutive functions is partial, the composed function is generally partial, however it may sometimes be total: for example  $succ: \mathbb{N} \to \mathbb{N}$  is total,  $pred: \mathbb{N} \to \mathbb{N}$  is partial, however  $pred \circ succ: \mathbb{N} \to \mathbb{N}$  is total. Hence we should have  $succ \mapsto tot$ ,  $pred \mapsto part$  and  $pred \circ succ \mapsto tot$ ; but  $part \circ tot = part \neq tot$ , so that this cannot define an ambifunctor.

A similar problem arises when we try to extend the indexation  $h_{s,p}^0$  to the following ambigraph  $\mathcal{G}_{s,p}$ :

Ambigraph  $\mathcal{G}_{s,p}$ : extends:  $\mathcal{G}_{s,p}^{0}$ , equation:  $p \circ s \equiv id_N$ .

Let  $h_{s,p}: \mathcal{G}_{s,p} \to \mathcal{I}_{part}$  be an ambifunctor extending  $h_{s,p}^0$ . Since  $h_{s,p}^0(N) = I$ , we must have  $h_{s,p}(id_N) = id_I = tot$ . And since  $h_{s,p}^0(s) = tot$  and  $h_{s,p}^0(p) = part$ , we must have  $h_{s,p}(p \circ s) = part \circ tot = part$ . So it is impossible to define  $h_{s,p}(p \circ s) = id_N$  in a way compatible with the projections of both members of the equation. In other words, it is impossible to extend  $h_{s,p}^0$  into an ambifunctor  $h_{s,p}: \mathcal{G}_{s,p} \to \mathcal{I}_{part}$ .

Both these problems come from the fact that by composing a total function and a partial function we generally get a partial function, but not always. This last point is not expressed by the composition  $part \circ tot = part$  in  $\mathcal{I}_{part}$ , and the composition  $part \circ tot = tot$  would still be worse. We will see in [guide2] how it is possible to overcome this difficulty.

#### 4.2 Blow-ups

We now come to the definition of the projective sketch  $\mathbf{E} \setminus \mu$ , blow-up of  $\mathbf{E}$  by  $\mu$ .

For this purpose, we first define the compositive graph  $\underline{\mathbf{E}} \setminus \mu$ , where each point E of  $\mathbf{E}$  is counted as many times as there are ingredients of nature E in  $\mu$ : for all point E of  $\mathbf{E}$ , all ingredient x of  $\mu$  of nature E gives rise to a point [E, x]; and for all arrow  $e: E \to E'$  of  $\mathbf{E}$ , the map  $\mu(e): \mu(E) \to \mu(E')$  is replaced by its sagittal diagram.

**Definition 11** The blow-up of  $\underline{\mathbf{E}}$  by  $\mu$  is the following compositive graph  $\underline{\mathbf{E}} \setminus \mu$ :

- its points are the [E, x] where E is a point of  $\mathbf{E}$  and  $x \in \mu(E)$ ;
- its arrows (resp. its identity arrows) are the  $[e, x] : [E, x] \to [E', x']$  where  $e : E \to E'$  is an arrow (resp. an identity arrow) of  $\mathbf{E}$ ,  $x \in \mu(E)$  and  $x' = \mu(e)(x) \in \mu(E')$ ;
- a pair of consecutive arrows ( $[e_1, x_1], [e_2, x_2]$ ) is composable in  $\underline{\mathbf{E}} \setminus \mu$  if and only if the pair of arrows  $(e_1, e_2)$  is composable in  $\mathbf{E}$ , and then  $[e_2, x_2] \circ [e_1, x_1] = [e_2 \circ e_1, x_1]$ .

In the notation ( $[e_1, x_1], [e_2, x_2]$ ) for a pair of consecutive arrows,  $x_2$  is redundant, since  $x_2 = \mu(e_1)(x_1)$ . This is why we denote:

$$[(e_1, e_2), x_1] = ([e_1, x_1], [e_2, x_2])$$
.

Let  $\chi : \mathcal{C}_p(\mathcal{B}) \to \underline{\mathbf{E}}$  be a  $\mathcal{B}$ -projective cone of  $\mathbf{E}$ . For all element x of  $\mu(\chi(C))$  (where C is the vertex of  $\mathcal{C}_p(\mathcal{B})$ ) let us consider the  $\mathcal{B}$ -projective cone  $[\chi, x]$  of  $\underline{\mathbf{E}} \setminus \mu$ , defined as follows:

- the image of the vertex C of  $\mathcal{C}$  is the point  $[\chi(C), x]$  of  $\underline{\mathbf{E}} \setminus \mu$ ,
- the image of a point B of  $\mathcal{B}$  is the point  $[\chi(B), \chi(p_B)(x)]$  of  $\underline{\mathbf{E}} \setminus \mu$ ,
- the image of an arrow  $b: B \to B'$  of  $\mathcal{B}$  is the arrow  $[\chi(b), \chi(p_B)(x)]$  of  $\underline{\mathbf{E}} \setminus \mu$ ,
- and the image of a projection arrow  $p_B: C \to B$  of  $\mathcal{C}$  is the arrow  $[\chi(p_B), x]$  of  $\underline{\mathbf{E}} \setminus \mu$ .

**Definition 12** The *blow-up* of **E** by  $\mu$  is the projective sketch  $\mathbf{E} \setminus \mu$ , with support the compositive graph  $\underline{\mathbf{E}} \setminus \mu$  and with distinguished projective cones the  $[\chi, x]$ , where  $\chi$  is a distinguished projective cone of **E** and  $x \in \mu(\chi(C))$ .

The fibration:

$$\mathbf{E} \backslash \mu : \mathbf{E} \backslash \mu \to \mathbf{E}$$

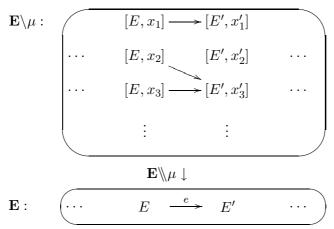
is the homomorphism of projective sketches defined as:

- $[E, x] \mapsto E$  for all point E of **E** and all  $x \in \mu(E)$ ;
- $[e, x] \mapsto e$  for all arrow e of  $\mathbf{E}$  and all  $x \in \mu(\mathtt{dom}(e))$  (note that the domain of an arrow e of  $\mathbf{E}$  is often denoted  $\mathtt{dom}(e)$  rather than  $\mathbf{E}(\mathtt{dom})(e)$ ).

Consequently the fibration satisfies:

•  $[(e_1, e_2), x_1] \mapsto (e_1, e_2)$  for all pair of consecutive arrows  $[(e_1, e_2), x_1]$  of **E** and all  $x \in \mu(\text{dom}(e_1))$ .

For example, if  $\mu(E) = \{x_1, x_2, x_3, \ldots\}$  and  $\mu(E') = \{x'_1, x'_2, x'_3, \ldots\}$  and  $\mu(e)$  is such that  $x_1 \mapsto x'_1, x_2 \mapsto x'_3$  and  $x_3 \mapsto x'_3$ , then:



Let  $h: \mu \to \mu'$  be a homomorphism of models of **E**. The homomorphism of projective sketches  $\mathbf{E} \setminus h : \mathbf{E} \setminus \mu \to \mathbf{E} \setminus \mu'$  is defined by  $(\mathbf{E} \setminus h)([e,x]) = [e,h(x)]$ . It is easy to check that in this way we get a functor:

$$\mathbf{E} \backslash -: \mathcal{M}od(\mathbf{E}) \to \mathcal{P}sk$$
.

**Example 15** The blow-up of a projective sketch generalizes the usual sagittal diagram of a map. Indeed, for all map  $f: X \to X'$ , the blow-up  $\mathbf{E}_{\mathcal{M}ap} \setminus \mathcal{I}_f$  of  $\mathbf{E}_{\mathcal{M}ap}$  by  $\mathcal{I}_f$  (see example 12) is precisely the sagittal diagram of f:

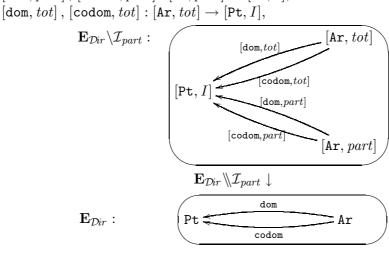
Projective sketch  $\mathbf{E}_{\mathcal{M}ap} \setminus \mathcal{I}_f$ :

points: [E, x] for all  $x \in X$ , [E', x'] for all  $x' \in X'$ , arrows:  $[e, x] : [E, x] \to [E', f(x)]$  for all  $x \in X$ .

**Example 16** Here is the blow-up of  $\mathbf{E}_{Dir}$  by  $\mathcal{I}_{part}^0$  (see example 13):

Projective sketch  $\mathbf{E}_{\mathcal{D}ir} \setminus \mathcal{I}_{part}^0$ :

 $\begin{array}{ll} points \colon & [\texttt{Pt},I], \, [\texttt{Ar},part], \, [\texttt{Ar},tot], \\ arrows \colon & [\texttt{dom},part], \, [\texttt{codom},part] \colon [\texttt{Ar},part] \to [\texttt{Pt},I], \end{array}$ 



We now come to the fundamental property of the blow-up:

Theorem 1 (Fundamental property of the blow-up) The category of indexations by  $\mu$  is equivalent to the category of models of the blow-up of  $\mathbf{E}$  by  $\mu$ :

$$\mathcal{M}od(\mathbf{E})/\mu \simeq \mathcal{M}od(\mathbf{E}\backslash\mu)$$
.

First, let us introduce some notations. On the points of both categories, the equivalence is denoted:

$$\begin{array}{cccc} \mathcal{M}od(\mathbf{E})/\mu & \simeq & \mathcal{M}od(\mathbf{E}\backslash\mu) \\ h & \stackrel{\mu/\!\!\backslash\!\!\backslash-}{\longmapsto} & \mu/\!\!\backslash\!\!\backslash h \\ \mu/\!\!\backslash\!\!\backslash \tau & \stackrel{\mu/\!\!\backslash\!\!\backslash-}{\longleftarrow} & \tau \; . \end{array}$$

Moreover, similarly to the notation  $\mathbf{E} \setminus \mu = \text{dom}(\mathbf{E} \setminus \mu)$ , we denote  $\mu \setminus \tau = \text{dom}(\mu \setminus \tau)$ , in such a way that:

$$\mu \mathbb{V} \tau : \mu \mathbb{V} \tau \to \mu$$
.

The theorem states that for all  $\tau$  (model of  $\mathbf{E} \setminus \mu$ ):

$$\mu / \! \backslash (\mu \backslash \! / \tau) \cong \tau$$
,

and that for all  $h: \nu \to \mu$  (indexation by  $\mu$ ):

$$\mu \mathbb{V}/(\mu \mathbb{N}h) \cong h$$
,

so that, looking at the domains of both homomorphisms:  $\mu \bigvee (\mu / \backslash h) \cong \nu$ .

About the proof. For all point of  $\mathcal{M}od(\mathbf{E})/\mu$ , i.e. all homomorphism  $h: \nu \to \mu$  of models of  $\mathbf{E}$ , we define the model  $\mu/\backslash h$  of  $\mathbf{E}\backslash \mu$  as follows:

• the interpretation of a point [E, x] of  $\mathbf{E} \setminus \mu$  is a subset of  $\nu(E)$ ; it is made up of the elements of  $\nu(E)$  which are mapped to x by h(E):

$$(\mu \wedge h)([E, x]) = \{ y \in \nu(E) \mid h(E)(y) = x \} \qquad (\subseteq \nu(E)) \quad ;$$

• and the interpretation of an arrow [e, x] of  $\mathbf{E} \setminus \mu$  is the restriction of the map  $\nu(e)$ :

$$(\mu \wedge h)([e, x])(y) = \nu(e)(y)$$
 for all  $y \in (\mu \wedge h)([E, x])$ .

In the opposite direction, for all model  $\tau$  of  $\mathbf{E} \setminus \mu$ , we define the homomorphism  $\mu \bigvee \tau : \mu \bigvee \tau \to \mu$  of models of  $\mathbf{E}$  by the following maps  $(\mu \bigvee \tau)(E) : (\mu \bigvee \tau)(E) \to \mu(E)$ , for all point E of  $\mathbf{E}$ :

• the set  $(\mu \lor \tau)(E)$  is the disjoint union of the sets  $\tau([E, x])$  for  $x \in \mu(E)$ :

$$(\mu \bigvee \tau)(E) = \sqcup_{x \in \mu(E)} \tau([E, x]) \qquad (\supseteq \tau([E, x])) \quad ;$$

• and the map  $(\mu \bigvee \tau)(E) : (\mu \bigvee \tau)(E) \to \mu(E)$  is defined piecewise:

$$(\mu \mathbb{V}/\tau)(E)(y) = x$$
 for all  $y \in \tau([E, x])$ .

The end of the proof is easy to check.

 $\Diamond$ 

Let  $h_1$  and  $h_2$  be two indexations by  $\mu$  in  $\mathcal{M}od(\mathbf{E})$  and let  $k:h_1 \to h_2$  be a  $\mu$ -triangle. It is easy to check that for all point E of  $\mathbf{E}$ , the map  $k(E):\nu_1(E) \to \nu_2(E)$  assigns to each element of  $(\mu/\!\!\backslash h_1)([E,x])$  an element of  $(\mu/\!\!\backslash h_2)([E,x])$ : indeed, let  $y_1 \in \nu_1(E)$  be such that  $h_1(E)(y_1) = x$ , then  $h_2(E)(k(E)(y_1)) = h_1(E)(y_1) = x$ . Let  $(\mu/\!\!\backslash k)([E,x])$  denote this restriction of the map k(E), it defines a homomorphism of models of  $\mathbf{E}\backslash \mu$ :

$$\mu / \! \backslash k : \mu / \! \backslash h_1 \to \mu / \! \backslash h_2$$
,

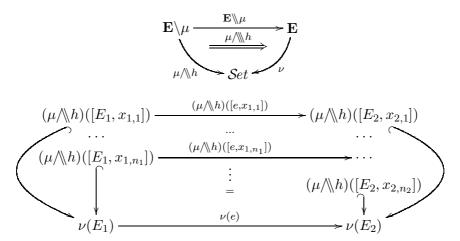
and a functor:

$$\mu/\!\!\! \backslash -: \mathcal{M}od(\mathbf{E})/\mu \to \mathcal{M}od(\mathbf{E}\backslash \mu)$$
.

On the other hand, let:

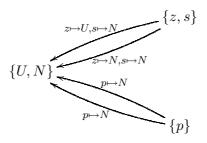
$$\mu / \! \backslash \! \backslash h : \mu / \! \backslash h \to \nu \circ (\mathbf{E} \backslash \! \backslash \mu)$$

denote the homomorphism of models of  $\mathbf{E} \setminus \mu$  defined by the inclusion  $(\mu / h)([E, x]) \subseteq \nu(E)$  for all point [E, x] of  $\mathbf{E} \setminus \mu$ . Then  $\mu / h = \text{dom}(\mu / h)$ .



**Example 17** Let us consider the indexation  $h_{part}^0$  of  $\mathcal{F}unc$  by  $\mathcal{I}_{part}^0$  as in example 13. The model  $\mathcal{I}_{part}^0 / h_{part}^0$  of  $\mathbf{E}_{\mathcal{D}ir} / \mathcal{I}_{part}^0$  interprets the points [Pt, I], [Ar, part] and [Ar, tot] respectively as sets, partial functions and total functions. It interprets the arrows [dom, part] and [codom, part] (resp. [dom, tot] and [codom, tot]) as the domain and codomain maps restricted to partial (resp. total) functions.

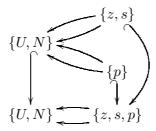
Similarly, the indexation  $h_{s,p}^0: \mathcal{G}_{s,p}^0 \to \mathcal{I}_{part}^0$  of the directed graph  $\mathcal{G}_{s,p}^0$  by  $\mathcal{I}_{part}^0$  from example 13 determines the following model  $\mathcal{I}_{part}^0 / h_{s,p}^0$  of  $\mathbf{E}_{\mathcal{D}ir} / \mathcal{I}_{part}^0$ :



The homomorphism of models of  $\mathbf{E}_{Dir} \setminus \mathcal{I}_{part}^0$ :

$$\mathcal{I}_{part}^{0}/\!\!\backslash\!\!\backslash h_{s,p}^{0}:\mathcal{I}_{part}^{0}/\!\!\backslash\!\!\backslash h_{s,p}^{0}\to\mathcal{G}_{s,p}^{0}\circ(\mathbf{E}_{\mathcal{D}ir}\backslash\!\!\backslash\mathcal{I}_{part}^{0})$$

is defined by the inclusions:



#### 4.3 Indexations of wefts

The definition of a  $\mu$ -indexation can be generalized to the wefts of models of  $\mathbf{E}$ , in such a way that the category of  $(\mathcal{M}od(\mathbf{E})/\mu)$ -wefts can be identified to the category of  $\mathcal{M}od(\mathbf{E})$ -wefts together with a  $\mu$ -indexation.

An indexation H of a  $\mathcal{M}od(\mathbf{E})$ -weft **S** by  $\mu$ , as defined below, is denoted:

$$H: \mathbf{S} \to \mu$$

though **S** and  $\mu$  do not belong to the same category:  $\mu$  is a point of  $\mathcal{M}od(\mathbf{E})$ , whereas **S** is a point of  $\mathcal{W}eft(\mathcal{M}od(\mathbf{E}))$ .

The definition of an indexation by  $\mu$  of a weft **S** of models of **E** is made recursively on the level of **S**, together with the definition of the composition  $H \circ \sigma$  of an indexation  $H : \mathbf{S} \to \mu$  by a homomorphism of wefts  $\sigma : \mathbf{S}' \to \mathbf{S}$ .

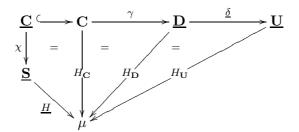
**Definition 13** If **S** has level 0, an *indexation* H *of* **S** *by*  $\mu$  is an indexation  $\underline{H}$  of  $\underline{\mathbf{S}}$  by  $\mu$ . The composition  $H \circ \sigma$  of an indexation by a homomorphism of wefts  $\sigma : \mathbf{S}' \to \mathbf{S}$  of level 0 is the composition  $\underline{H} \circ \underline{\sigma}$  as homomorphisms of models of  $\mathbf{E}$ .

If **S** has level  $n \ge 1$ , an indexation H of **S** by  $\mu$  is an indexation  $\underline{H}$  of  $\underline{\mathbf{S}}$  by  $\mu$  extended to all the constraints of **S**: for all constraint  $\Gamma = (\underline{\mathbf{C}} \xrightarrow{\chi} \underline{\mathbf{S}}, \mathbf{C} \xrightarrow{\gamma} \mathbf{D} \xrightarrow{\delta} \mathbf{U})$  of **S**, an extension of  $\underline{H}$  to  $\Gamma$  is made up of three indexations  $H_{\mathbf{C}}$ ,  $H_{\mathbf{D}}$  and  $H_{\mathbf{U}}$  of respectively **C**,  $\mathbf{D}$  and  $\mathbf{U}$  by  $\mu$ , such that:

$$\underline{H} \circ \chi = \underline{H}_{\mathbf{C}}, H_{\mathbf{D}} \circ \gamma = H_{\mathbf{C}}, H_{\mathbf{U}} \circ \delta = H_{\mathbf{D}}.$$

The composition  $H \circ \sigma$  of an indexation by a homomorphism of wefts  $\sigma : \mathbf{S}' \to \mathbf{S}$  of level  $\leq n$  is made up of the indexation  $\underline{H} \circ \underline{\sigma} = \underline{H} \circ \underline{\sigma}$  of  $\underline{\mathbf{S}}'$  by  $\mu$  extended to all the constraints of  $\mathbf{S}'$  in the following way:

let  $\Gamma' = (\underline{\mathbf{C}} \xrightarrow{\chi} \underline{\mathbf{S}}', \mathbf{C} \xrightarrow{\gamma} \mathbf{D} \xrightarrow{\delta} \mathbf{U})$  be a constraint of  $\mathbf{S}'$ , then from the definition of a homomorphism of wefts  $(\underline{\mathbf{C}} \xrightarrow{\underline{\sigma} \circ \chi} \underline{\mathbf{S}}, \mathbf{C} \xrightarrow{\gamma} \mathbf{D} \xrightarrow{\delta} \mathbf{U})$  is a constraint of  $\mathbf{S}$ . Now let  $(H \circ \sigma)_{\mathbf{C}} = H_{\mathbf{C}}$ ,  $(H \circ \sigma)_{\mathbf{D}} = H_{\mathbf{D}}$  and  $(H \circ \sigma)_{\mathbf{U}} = H_{\mathbf{U}}$ .



Then we may define the category of indexations by  $\mu$  in  $Weft(Mod(\mathbf{E}))$ :

$$Weft(Mod(\mathbf{E}))/\mu$$
.

Its points are the indexations by  $\mu$  of  $\mathcal{M}od(\mathbf{E})$ -wefts, and an arrow from  $H_1: \mathbf{S}_1 \to \mu$  to  $H_2: \mathbf{S}_2 \to \mu$  is a homomorphism of  $\mathcal{M}od(\mathbf{E})$ -wefts  $K: \mathbf{S}_1 \to \mathbf{S}_2$  such that  $H_2 \circ K = H_1$ .

$$\mathbf{S}_1 \xrightarrow{K} \mathbf{S}_2$$

$$= H_2$$

Now it is easy to check the equivalence:

#### **Proposition 4**

$$Weft(Mod(\mathbf{E}))/\mu \simeq Weft(Mod(\mathbf{E})/\mu)$$
.

From the equivalence in theorem 1 follows the equivalence of the categories of wefts:

$$Weft(Mod(\mathbf{E})/\mu) \simeq Weft(Mod(\mathbf{E}\backslash\mu))$$
,

whence, by proposition 4, the equivalence:

$$Weft(Mod(\mathbf{E}))/\mu \simeq Weft(Mod(\mathbf{E}\backslash\mu))$$
,

which is denoted:

$$H \mapsto \mu / \! \backslash H$$
.

Let **S** be a  $\mathcal{M}od(\mathbf{E})$ -weft and  $H: \mathbf{S} \to \mu$  a  $\mu$ -indexation of **S**. The homomorphism of models of  $\mathbf{E} \setminus \mu$ :

$$\mu / \! \backslash \! \underline{H} : \mu / \! \backslash \underline{H} \to \underline{\mathbf{S}} \circ (\mathbf{E} \backslash \! \backslash \mu)$$

has been defined in 4.2 from the inclusions  $(\mu / \backslash \underline{H})([E, x]) \subseteq \underline{\mathbf{S}}(E)$  for all point [E, x] of  $\mathbf{E} \backslash \mu$ . It can easily be extended, recursively on the level of  $\mathbf{S}$ , into:

$$\mu / \! \backslash \! H : \mu / \! \backslash H \to \mathbf{S} \circ (\mathbf{E} \backslash \! \backslash \mu)$$

which, however, is not a homomorphism of  $\mathcal{M}od(\mathbf{E}\backslash\mu)$ -wefts. It is a loose homomorphism, as defined below.

$$\begin{array}{c|c}
\mathbf{E} & & \mathbf{E} & \\
\mu & & \mathbf{E} & \\
\downarrow \mu / \mathbb{N} & & \mathbf{E} \\
\downarrow \rho / \mathbb{N} & & \mathbf{S} \\
\end{array}$$

**Definition 14** Let  $\mathcal{A}$  be a category and  $\mathbf{S}$  and  $\mathbf{S}'$  two  $\mathcal{A}$ -wefts. A loose homomorphism  $\widetilde{\sigma}: \mathbf{S} \to \mathbf{S}'$  of  $\mathcal{A}$ -wefts is made up of an arrow  $\underline{\widetilde{\sigma}}: \underline{\mathbf{S}} \to \underline{\mathbf{S}}'$  of  $\mathcal{A}$  and, for all constraint of  $\mathbf{S}$ :

$$\Gamma = (\underline{\mathbf{C}} \xrightarrow{\chi} \underline{\mathbf{S}}, \mathbf{C} \xrightarrow{\gamma} \mathbf{D} \xrightarrow{\delta} \mathbf{U}),$$

of a constraint of S':

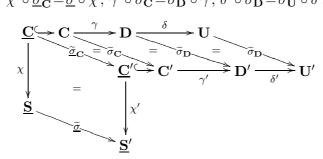
$$\Gamma' = (\underline{\mathbf{C}}' \xrightarrow{\chi'} \underline{\mathbf{S}}', \underline{\mathbf{C}}' \xrightarrow{\gamma'} \underline{\mathbf{D}}' \xrightarrow{\delta'} \underline{\mathbf{U}}'),$$

and three loose homomorphisms:

$$\widetilde{\sigma}_{\mathbf{C}}: \mathbf{C} \to \mathbf{C}', \ \widetilde{\sigma}_{\mathbf{D}}: \mathbf{D} \to \mathbf{D}', \ \widetilde{\sigma}_{\mathbf{U}}: \mathbf{U} \to \mathbf{U}',$$

such that (with the natural definition of composition):

$$\chi' \circ \underline{\widetilde{\sigma}}_{\mathbf{C}} = \underline{\widetilde{\sigma}} \circ \chi \,, \, \gamma' \circ \widetilde{\sigma}_{\mathbf{C}} = \widetilde{\sigma}_{\mathbf{D}} \circ \gamma \,, \, \delta' \circ \widetilde{\sigma}_{\mathbf{D}} = \widetilde{\sigma}_{\mathbf{U}} \circ \delta \,.$$



This definition can be compared with the definition of the homomorphisms of  $\mathcal{A}$ -wefts, given in 2.4. It follows that a homomorphism of  $\mathcal{A}$ -wefts is a loose homomorphism such that, for all constraint  $\Gamma$ , we have  $\Gamma = \Gamma'$  and the three loose homomorphisms  $\widetilde{\sigma}_{\mathbf{C}}$ ,  $\widetilde{\sigma}_{\mathbf{D}}$  and  $\widetilde{\sigma}_{\mathbf{U}}$  are the identities.

The homomorphisms of  $\mathcal{A}$ -wefts behave well with respect to the realizations: let  $\sigma: \mathbf{S} \to \mathbf{S}'$  be a homomorphism of  $\mathcal{A}$ -wefts, if an arrow  $\omega': \underline{\mathbf{S}}' \to A$  satisfies the constraints of  $\mathbf{S}'$ , then the arrow  $\omega' \circ \underline{\sigma}: \underline{\mathbf{S}} \to A$  satisfies the constraints of  $\mathbf{S}$ . Conversely, the loose homomorphisms of  $\mathcal{A}$ -wefts do *not* behave well with respect to the realizations. However, this is not always a drawback. For instance, in [guide2], this is the kind of property which we will be interested in.

**Example 18** Let us consider the projective sketch  $\mathbf{E}_{Dir}$  and the directed graph  $\mathcal{I}_{part}^0$  from example 13. Proposition 4 states that:

$$Weft(Dir)/\mathcal{I}_{part}^0 \simeq Weft(Dir/\mathcal{I}_{part}^0)$$
.

We now look at this isomorphism by an example.

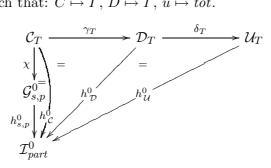
Let us consider the following weft of directed graphs  $\mathbf{S}_{s,p}^0$ , with support  $\underline{\mathbf{S}}_{s,p}^0 = \mathcal{G}_{s,p}^0$  (see example 13) and with the unique constraint  $U = \mathbb{I}$ :

 $\mathcal{D}ir$ -Weft  $\mathbf{S}_{s,p}^0$ :

support:  $\mathcal{G}_{s,p}^0$ , terminal point: U.

Let us consider the indexation  $h_{s,p}^0: \mathcal{G}_{s,p}^0 \to \mathcal{I}_{part}^0$  from example 13 as well as the indexations:

- $h^0_{\mathcal{C}}: \mathcal{C}_T \to \mathcal{I}^0_{part}$  such that:  $C \mapsto I$ ,
- $h_{\mathcal{D}}^0: \mathcal{D}_T \to \mathcal{I}_{part}^0$  such that:  $C \mapsto I, D \mapsto I$ ,
- $h^0_{\mathcal{U}}: \mathcal{U}_T \to \mathcal{I}^0_{part}$  such that:  $C \mapsto I, D \mapsto I, u \mapsto tot$ .



Then  $h_{s,p}^0$  extended by  $(h_{\mathcal{C}}^0, h_{\mathcal{D}}^0, h_{\mathcal{U}}^0)$  makes up an indexation  $H_{s,p}^0$  of  $\mathbf{S}_{s,p}^0$  by  $\mathcal{I}_{part}^0$ . Hence,

 $H^0_{s,p}$  is a point of the category  $Weft(\mathcal{D}ir)/\mathcal{I}^0_{part}$ . Now, let us look at  $h^0_{s,p}$ ,  $h^0_{\mathcal{C}}$ ,  $h^0_{\mathcal{D}}$  et  $h^0_{\mathcal{U}}$  as four  $\mathcal{I}^0_{part}$ -indexations. The homomorphisms of directed graphs  $\chi: \mathcal{C}_T \to \mathcal{G}^0_{s,p}$ ,  $\gamma_T: \mathcal{C}_T \to \mathcal{D}_T$  and  $\delta_T: \mathcal{D}_T \to \mathcal{U}_T$  give rise to  $\mathcal{I}^0_{part}$ -triangles  $\chi: h^0_{\mathcal{C}} \to h^0_{s,p}$ ,  $\gamma_T: h^0_{\mathcal{C}} \to h^0_{\mathcal{D}}$  and  $\delta_T: h^0_{\mathcal{D}} \to h^0_{\mathcal{U}}$ . In this way we get a weft of  $\mathcal{I}^0_{part}$ -indexations, with support  $h^0_{s,p}$  and with constraint  $(h^0_{\mathcal{C}} \xrightarrow{\chi} h^0_{s,p}, h^0_{\mathcal{C}} \xrightarrow{\gamma_T} h^0_{\mathcal{D}} \xrightarrow{\delta_T} h^0_{\mathcal{U}})$ : this is a point of the category  $Weft(\mathcal{D}ir/\mathcal{I}^0_{part})$ . It corresponds, by the equivalence of proposition 4, to the point  $H_{s,p}^0$  of the category  $Weft(Dir)/\mathcal{I}_{part}^0$ .

The indexation  $H^0_{s,p}$  of  $\mathbf{S}^0_{s,p}$  by  $\mathcal{I}^0_{part}$  gives rise to the  $\mathcal{M}od(\mathbf{E}_{\mathcal{D}ir}\backslash\mathcal{I}^0_{part})$ -weft  $\mathcal{I}^0_{part}/\mathcal{M}H^0_{s,p}$ . Its support is  $\mathcal{I}^0_{part}/\mathcal{M}h^0_{s,p}$  (see example 17). In order to describe its unique constraint, let us first consider the models  $\mathcal{I}^0_{part}/\mathcal{M}h^0_{...}$  of  $\mathbf{E}_{\mathcal{D}ir}\backslash\mathcal{I}^0_{part}$ , where "..." means  $\mathcal{C}$ ,  $\mathcal{D}$  or  $\mathcal{U}$  (see example 18). They are quite simple:  $(\mathcal{I}^0_{part}/\mathcal{M}h^0_{...})([E,x]) = \emptyset$  except for:

- $(\mathcal{I}_{nart}^0 / h_C^0)([Pt, I]) = \{C\},$
- $(\mathcal{I}_{nart}^0 \wedge h_{\mathcal{D}}^0)([\mathsf{Pt},I]) = \{C,D\},$
- $(\mathcal{I}_{nart}^0 / h_{\mathcal{U}}^0)([\mathsf{Pt}, I]) = \{C, D\} \text{ and } (\mathcal{I}_{nart}^0 / h_{\mathcal{U}}^0)([\mathsf{Ar}, tot]) = \{u\}.$

The unique constraint of the  $\mathcal{M}od(\mathbf{E}_{\mathcal{D}ir} \setminus \mathcal{I}^0_{part})$ -weft  $\mathcal{I}^0_{part} / \backslash H^0_{s,p}$  is denoted  $\mathcal{I}^0_{part} / \backslash \Gamma_T$ . Its body is:

$$\mathcal{I}_{part}^{0} / \! \! \backslash \! \! \backslash h_{\mathcal{C}}^{0} \xrightarrow{\mathcal{I}_{part}^{0} / \! \! \backslash \chi} \mathcal{I}_{part}^{0} / \! \! \backslash h_{s,p}^{0}$$

defined by:

$$(\mathcal{I}^0_{part}/\!\!\backslash\chi)([\mathtt{Pt},I]):C\mapsto U$$
 .

Its potential is:

$$\mathcal{I}^{0}_{part} / \! \! \backslash h^{0}_{\mathcal{C}} \xrightarrow{\mathcal{I}^{0}_{part} / \! \! \backslash \gamma_{T}} \mathcal{I}^{0}_{part} / \! \! \backslash h^{0}_{\mathcal{D}} \xrightarrow{\mathcal{I}^{0}_{part} / \! \! \backslash \delta_{T}} \mathcal{I}^{0}_{part} / \! \! \backslash h^{0}_{\mathcal{U}}$$

where  $\mathcal{I}_{part}^0/\!\!\! \backslash \gamma_T$  and  $\mathcal{I}_{part}^0/\!\!\! \backslash \delta_T$  are defined by the inclusions (for all point [E,x] of  $\mathbf{E}_{Dir}\backslash \mathcal{I}_{part}^0$ ):

$$(\mathcal{I}^0_{part}/\!\!\! \backslash h^0_{\mathcal{C}})([E,x]) \subseteq (\mathcal{I}^0_{part}/\!\!\! \backslash h^0_{\mathcal{D}})([E,x]) \subseteq (\mathcal{I}^0_{part}/\!\!\! \backslash h^0_{\mathcal{U}})([E,x]) \ .$$

The loose homomorphism of  $\mathcal{M}od(\mathbf{E}_{\mathcal{D}ir} \setminus \mathcal{I}_{nart}^0)$ -wefts:

$$\mathcal{I}_{part}^0 / \! \backslash \! H_{s,p}^0 : \mathcal{I}_{part}^0 / \! \backslash \! H_{s,p}^0 \to \mathbf{S}_{s,p}^0 \circ (\mathbf{E}_{Dir} \backslash \! \backslash \mathcal{I}_{part}^0)$$

extends the homomorphism  $\mathcal{I}^0_{part}/\mathbb{N}h^0_{s,p}$  described in example 17. It is only a loose homomorphism phism of  $\mathcal{M}od(\mathbf{E}_{\mathcal{D}ir} \setminus \mathcal{I}_{part}^0)$ -wefts because, on one hand:

$$(\mathcal{I}^0_{part}/\!\!\backslash h^0_{\,\mathcal{U}})([\mathtt{Ar},\mathit{part}]) = \emptyset\,,$$

while, on the other hand:

$$(\mathbf{S}_{s,p}^0 \circ (\mathbf{E}_{\mathcal{D}\!ir} \backslash \! \backslash \mathcal{I}_{part}^0))([\mathtt{Ar},\mathit{part}]) = \mathbf{S}_{s,p}^0(\mathtt{Ar}) = \{u\}\,,$$

hence both constraints are different.

Finally, let us consider the realizations of this  $\mathcal{M}od(\mathbf{E}_{\mathcal{D}ir} \setminus \mathcal{I}_{nart}^0)$ -weft:

$$\mathcal{I}_{part}^0 / \! \backslash H_{s,p}^0 : \mathbf{E}_{Dir} \backslash \mathcal{I}_{part}^0 \xrightarrow{\frown} \mathcal{S}et$$

towards the point of  $\mathbf{E}_{Dir} \setminus \mathcal{I}_{part}^0$ :

$$\mathcal{I}_{part}^0 / h_{part}^0 : \mathbf{E}_{Dir} \backslash \mathcal{I}_{part}^0 \to \mathcal{S}et$$

(see example 17). By definition, these realizations are the homomorphisms of  $\mathbf{E}_{Dir} \setminus \mathcal{I}_{part}^0$ -models from  $\mathcal{I}_{part}^0 / h_{s,p}^0$  towards  $\mathcal{I}_{part}^0 / h_{part}^0$  which satisfy the constraint  $\mathcal{I}_{part}^0 / h_{s,p}^0$ .

It means that such a realization  $\omega$  is made up of:

- two sets  $\omega(U)$  and  $\omega(N)$ ,
- two total functions  $\omega(z):\omega(U)\to\omega(N)$  and  $\omega(s):\omega(N)\to\omega(N)$  and one partial function  $\omega(p):\omega(N)\to\omega(N)$ ,
- and in addition the set  $\omega(U)$  is such that for all set X there is a unique total function  $f: X \to \omega(U)$ : it means that  $\omega(U)$  is a one-element set.

#### 4.4 Yoneda counter-model

The Yoneda functor is considered here in the framework of projective sketches, see [Lair 71] and the reference manual.

Let  $\mathbf{E}$  be a projective sketch. For all point E of  $\mathbf{E}$ , let  $\mathcal{Y}_{\mathbf{E}}(E)$  be the model of  $\mathbf{E}$  freely generated by an ingredient of nature E (it can be proven that such a model does exist). Then to each arrow  $e: E \to E'$  of  $\mathbf{E}$  is canonically associated a homomorphism of  $\mathbf{E}$ -models  $\mathcal{Y}_{\mathbf{E}}(e): \mathcal{Y}_{\mathbf{E}}(E') \to \mathcal{Y}_{\mathbf{E}}(E)$ . This defines a contravariant functor  $\mathcal{Y}_{\mathbf{E}}$  from  $\underline{\mathbf{E}}$  to  $\mathcal{M}od(\mathbf{E})$ . In addition, this contravariant functor maps each distinguished projective cone of  $\mathbf{E}$  to a limit inductive cone of  $\mathcal{M}od(\mathbf{E})$ : for this reason, it is called a *counter-model*. It is the Yoneda counter-model of  $\mathbf{E}$ :

$$\mathcal{Y}_{\mathbf{E}}: \mathbf{E} \longrightarrow \mathcal{M}od(\mathbf{E})$$
.

**Example 19** Let  $\mathbf{E} = \mathbf{E}_{\mathcal{M}ap}$  as in example 12. Then:

- the model  $\mathcal{Y}_{\mathbf{E}}(E)$  of  $\mathbf{E}_{\mathcal{M}ap}$  is made up of two sets  $\mathcal{Y}_{\mathbf{E}}(E)(E) = \{x\}$  and  $\mathcal{Y}_{\mathbf{E}}(E)(E') = \{x'\}$  and of the map  $\mathcal{Y}_{\mathbf{E}}(E)(e) : x \mapsto x'$ ;
- the model  $\mathcal{Y}_{\mathbf{E}}(E')$  of  $\mathbf{E}_{\mathcal{M}ap}$  is made up of two sets  $\mathcal{Y}_{\mathbf{E}}(E')(E) = \emptyset$  and  $\mathcal{Y}_{\mathbf{E}}(E')(E') = \{y'\}$  and of the unique map  $\mathcal{Y}_{\mathbf{E}}(E')(e)$  of  $\emptyset$  to  $\{y'\}$ ;
- the homomorphism  $\mathcal{Y}_{\mathbf{E}}(e): \mathcal{Y}_{\mathbf{E}}(E') \to \mathcal{Y}_{\mathbf{E}}(E)$  is made up of the unique maps  $\mathcal{Y}_{\mathbf{E}}(e)(E): \emptyset \to \{x\}$  and  $\mathcal{Y}_{\mathbf{E}}(e)(E'): \{y'\} \to \{x'\}.$

**Example 20** Let us consider the projective sketch  $\mathbf{E}_{\mathcal{A}mbi}$  described in 3.4, and let  $\mathcal{Y} = \mathcal{Y}_{\mathbf{E}_{\mathcal{A}mbi}}$ . Proposition 3 proves that  $\mathcal{M}od(\mathbf{E}_{\mathcal{A}mbi})$  is equivalent to  $\mathcal{A}mbi$ , hence:

$$\mathcal{Y} = \mathcal{Y}_{\mathbf{E}_{Ambi}} : \mathbf{E}_{Ambi} \longrightarrow \mathcal{A}mbi$$
.

The counter-model  $\mathcal{Y}$  interprets each point E of  $\mathbf{E}_{\mathcal{A}mbi}$  as an ambigraph  $\mathcal{Y}(E)$  and each arrow  $e: E \to E'$  of  $\mathbf{E}_{\mathcal{A}mbi}$  as an ambifunctor  $\mathcal{Y}(e): \mathcal{Y}(E') \to \mathcal{Y}(E)$ . For instance:

- the ambigraph  $\mathcal{Y}(Pt)$  is made of a single point, denoted Y;
- the ambigraph  $\mathcal{Y}(Ar)$  is made of two points  $Y_d$  and  $Y_c$  and one arrow  $y:Y_d\to Y_c$ ;
- the ambifunctor  $\mathcal{Y}(\mathtt{dom}): \mathcal{Y}(\mathtt{Pt}) \to \mathcal{Y}(\mathtt{Ar})$  is defined by  $Y \mapsto Y_d$ ;
- the ambifunctor  $\mathcal{Y}(\mathtt{codom}): \mathcal{Y}(\mathtt{Pt}) \to \mathcal{Y}(\mathtt{Ar})$  is defined by  $Y \mapsto Y_c$ .

For all point E of  $\mathbf{E}_{\mathcal{A}mbi}$ , the following projective sketch can be obtained by a blow up:

$$\mathcal{Z}(E) = \mathbf{E}_{Ambi} \setminus (\mathcal{Y}(E))$$
.

Each  $\mathcal{Z}(E)$  is a projective sketch, as  $\mathbf{E}_{\mathcal{A}mbi}$ , and  $\mathcal{Z}$  defines a contravariant functor:

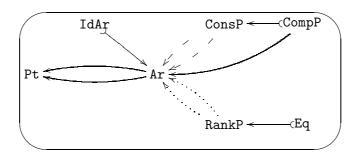
$$\mathcal{Z}: \mathbf{E}_{\mathcal{A}mbi} \longrightarrow \mathcal{P}sk$$
.

For instance:

- the projective sketch  $\mathcal{Z}(Pt)$  is made of a single point [Pt, Y];
- the projective sketch  $\mathcal{Z}(\mathtt{Ar})$  is made of three points  $[\mathtt{Pt},Y_d]$ ,  $[\mathtt{Pt},Y_c]$  and  $[\mathtt{Ar},y]$  and of two arrows  $[\mathtt{dom},y]:[\mathtt{Ar},y] \to [\mathtt{Pt},Y_d]$  and  $[\mathtt{codom},y]:[\mathtt{Ar},y] \to [\mathtt{Pt},Y_c];$
- the homomorphism  $\mathcal{Z}(\mathtt{dom}): \mathcal{Z}(\mathtt{Pt}) \to \mathcal{Z}(\mathtt{Ar})$  is defined by  $[\mathtt{Pt}, Y] \mapsto [\mathtt{Pt}, Y_d]$ ;
- the homomorphism  $\mathcal{Z}(\mathtt{codom}): \mathcal{Z}(\mathtt{Pt}) \to \mathcal{Z}(\mathtt{Ar})$  is defined by  $[\mathtt{Pt}, Y] \mapsto [\mathtt{Pt}, Y_c]$ .

More generally, the table below draws the ambigraph  $\mathcal{Y}(E)$  and the projective sketch  $\mathcal{Z}(E)$  for all point E of  $\mathbf{E}_{\mathcal{A}mbi}$ . For all arrow  $e: E \to E'$  of  $\mathbf{E}_{\mathcal{A}mbi}$ , it is then easy to determine the ambifunctor  $\mathcal{Y}(e): \mathcal{Y}(E') \to \mathcal{Y}(E)$  and the homomorphism  $\mathcal{Z}(e): \mathcal{Z}(E') \to \mathcal{Z}(E)$ .

Actually, in this table, only the directed graph sublying  $\mathcal{Z}(E)$  is drawn, however constraints are suggested in the following way: monomorphisms are drawn as  $\longrightarrow$  and the projections from both distinguished cones  $\Gamma_{\texttt{CompP}}$  and  $\Gamma_{\texttt{Eq}}$  are drawn as -- > and  $\longrightarrow$  respectively. With these conventions, the directed graph sublying  $\mathbf{E}_{\mathcal{A}mbi}$  is the following one:



Here is the table of the interpretation of the points of  $\mathbf{E}_{\mathcal{A}mbi}$  by  $\mathcal{Y} = \mathcal{Y}_{\mathbf{E}_{\mathcal{A}mbi}} : \mathbf{E}_{\mathcal{A}mbi} \longrightarrow \mathcal{A}mbi$  and by  $\mathcal{Z} : \mathbf{E}_{\mathcal{A}mbi} \longrightarrow \mathcal{P}sk$ :

E	$\mathcal{Y}(E)$	$\mathcal{Z}(E) = \mathbf{E}_{\mathcal{A}mbi} \backslash \mathcal{Y}(E)$
Pt	•	•
Ar	• — •	
IdAr	?	
ConsP	•> •	
CompP		
RankP	( )	
Eq		

It is easy to check that  $\mathcal{Y}$  is a counter-model of  $\mathbf{E}_{\mathcal{A}mbi}$ , and that  $\mathcal{Z}$  is not a counter-model of  $\mathbf{E}_{\mathcal{A}mbi}$ : for example the homomorphism  $\mathcal{Z}(\mathbf{j}_{\mathtt{IdAr}}):\mathcal{Z}(\mathtt{Ar})\to\mathcal{Z}(\mathtt{IdAr})$  is not an epimorphism.

Theorem 1 states that the categories  $\mathcal{M}od(\mathcal{Z}(\mathtt{Ar}))$  and  $\mathcal{M}od(\mathbf{E}_{\mathcal{A}mbi})/(\mathcal{Y}(\mathtt{Ar}))$  are equivalent. We now check directly that the set of models of  $\mathcal{Z}(\mathtt{Ar})$  is in one-to-one correspondence with the set of indexations of ambigraphs by  $\mathcal{Y}(\mathtt{Ar})$ .

On one hand, the models of  $\mathcal{Z}(\mathtt{Ar})$  are the pairs of maps with the same domain: they are called the *set-valued spans*.

On the other hand, the indexations of ambigraphs by  $\mathcal{Y}(\mathtt{Ar})$  are the ambifunctors  $h: \mathcal{G} \to \mathcal{Y}(\mathtt{Ar})$ . Such an indexation h is characterized by the three sets  $X_d = h(\mathtt{Pt})^{-1}(Y_d)$ ,  $X_c = h(\mathtt{Pt})^{-1}(Y_c)$  and  $X = h(\mathtt{Ar})^{-1}(y)$  (which satisfy  $\mathcal{G}(\mathtt{Pt}) = X_d \sqcup X_c$  and  $\mathcal{G}(\mathtt{Ar}) = X$ ) and the two maps  $\mathcal{G}(\mathtt{dom}), \mathcal{G}(\mathtt{codom}): X \to X_d \sqcup X_c$  such that  $\mathcal{G}(\mathtt{dom})(X) \subseteq X_d$  and  $\mathcal{G}(\mathtt{codom})(X) \subseteq X_c$ . Consequently, h is characterized by the set-valued span  $(X_d \overset{f_d}{\longleftarrow} X \overset{f_c}{\longrightarrow} X_c)$ , where  $f_d$  and  $f_c$  respectively denote the restrictions of  $\mathcal{G}(\mathtt{dom})$  and  $\mathcal{G}(\mathtt{codom})$ .

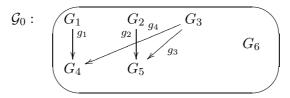
$$X_d \sqcup X_c = \mathcal{G}(\operatorname{Pt}) \xrightarrow{h(\operatorname{Pt})} \mathcal{Y}(\operatorname{Ar})(\operatorname{Pt}) = \{Y_d, Y_c\}$$

$$X_d \bigcup_{f_d} \mathcal{G}(\operatorname{dom}) \bigvee_{f_c} X_c \qquad \mathcal{Y}(\operatorname{Ar})(\operatorname{dom}) \bigvee_{f_d} (\operatorname{Ar})(\operatorname{codom})$$

$$X_c \longmapsto_{f_d} \mathcal{Y}(\operatorname{Ar})(\operatorname{Ar}) = \{y\}$$

The ambigraph  $\mathcal{G}_s$  from example 1 cannot be indexed by  $\mathcal{Y}(Ar)$ , since the point N is both a domain (for s) and a codomain (for z and s).

But the ambigraph  $\mathcal{G}_0$ :



can be indexed in two ways by  $\mathcal{Y}(Ar)$ : indeed all arrows of  $\mathcal{G}_0$  must be indexed by y, and the points  $G_1$ ,  $G_2$  and  $G_3$  (resp.  $G_4$  and  $G_5$ ) must be indexed by  $Y_d$  (resp.  $Y_c$ ) since they are domains (resp. codomains). But  $G_6$ , which is neither a domain nor a codomain, can be indexed either by  $Y_d$  or by  $Y_c$ . These indexations  $h_1$  and  $h_2$  correspond respectively to the set-valued spans:

$$\{G_1, G_2, G_3, G_6\} \underbrace{f_d} \\ \{g_1, g_2, g_3, g_4\} \\ \{G_4, G_5\} \underbrace{f_c} \\ \{g_1, g_2, g_3, g_4\} \\ \{G_4, G_5, G_6\} \underbrace{f_c} \\ \{g_1, g_2, g_3, g_4\}$$

The fundamental result on the Yoneda counter-model is the following theorem 2, proven in our reference manual. It is similar to the classical result on Yoneda functor.

The composition of the contravariant functors  $\mathcal{Y}_{\mathbf{E}}: \underline{\mathbf{E}} \longrightarrow \mathcal{M}od(\mathbf{E})$  and  $\mathrm{Hom}_{\mathcal{M}od(\mathbf{E})}(-,\mu): \mathcal{M}od(\mathbf{E}) \longrightarrow \mathcal{S}et$  gives rise to a functor  $\mathrm{Hom}_{\mathcal{M}od(\mathbf{E})}(\mathcal{Y}_{\mathbf{E}}(-),\mu): \underline{\mathbf{E}} \longrightarrow \mathcal{S}et$ . In addition, since  $\mathcal{Y}_{\mathbf{E}}$  is a counter-model of  $\mathbf{E}$  and  $\mathrm{Hom}_{\mathcal{M}od(\mathbf{E})}(-,\mu)$  is left-exact (which means that it maps each limit inductive cone of  $\mathcal{M}od(\mathbf{E})$  to a limit projective cone), the composed functor  $\mathrm{Hom}_{\mathcal{M}od(\mathbf{E})}(\mathcal{Y}_{\mathbf{E}}(-),\mu)$  is a model of  $\mathbf{E}$ :

$$\operatorname{Hom}_{\mathcal{M}od(\mathbf{E})}(\mathcal{Y}_{\mathbf{E}}(-), \mu) : \mathbf{E} \to \mathcal{S}et$$
.

$$\mathcal{M}od(\mathbf{E}) = \operatorname{Hom}_{\mathcal{M}od(\mathbf{E})}(\mathcal{Y}_{\mathbf{E}}(-), \mu)$$

$$\operatorname{Set}$$

Theorem 2 (Yoneda lemma for projective sketches) Let  $\mu$  be a model of E. Then:

$$\mu = IndLim(\underline{\mathbf{E}} \backslash \underline{\mu} \xrightarrow{\underline{\mathbf{E}} \backslash \mu} \underline{\underline{\mathbf{E}}} \xrightarrow{\mathcal{Y}_{\underline{\mathbf{E}}}} \mathcal{M}od(\underline{\mathbf{E}})),$$

and:

$$\mu \cong \operatorname{Hom}_{\mathcal{M}od(\mathbf{E})}(\mathcal{Y}_{\mathbf{E}}(-), \mu)$$
.

In the first part of this theorem, only the sublying contravariant functor of the counter-model  $\mathcal{Y}_{\mathbf{E}} \circ (\mathbf{E} \backslash \mu)$  of  $\mathbf{E} \backslash \mu$  is used to build the inductive limit.

**Example 21** The description of  $\mathcal{Y} = \mathcal{Y}_{\mathbf{E}_{\mathcal{A}mbi}}$  in example 20 gives the following interpretation of Yoneda lemma when  $\mathbf{E} = \mathbf{E}_{\mathcal{A}mbi}$ :

• the first part of theorem 2 states that each ambigraph  $\mathcal{G}$  can be built by taking each one of its ingredients (points, arrows, etc.) and gluing them together in the right way;

• the second part states that for all point E of  $\mathbf{E}_{\mathcal{A}mbi}$ , the ambigraph  $\mathcal{Y}(E)$  describes the shape of the ingredients of  $\mathcal{G}$  of nature E: the ambigraph  $\mathcal{Y}(\mathsf{Pt})$  describes the shape of the points of  $\mathcal{G}$ , the ambigraph  $\mathcal{Y}(\mathsf{Ar})$  describes the shape of the arrows of  $\mathcal{G}$ , etc.

Moreover, when  $\mathbf{E} = \mathbf{E}_{\mathcal{A}mbi}$  and  $\mu = \mathcal{S}et$ , the second part of theorem 2 states that:

$$Set \cong Hom_{Ambi}(\mathcal{Y}(-), Set)$$
.

This result is easy to check directly. Indeed it is clear (except for the problems related to the size of the sets, which are addressed in [ref1]) that:

- the models of  $\mathcal{Y}(Pt)$  are the sets,
- the models of  $\mathcal{Y}(Ar)$  are the maps,
- and so on...

5 CONCLUSION 43

## 5 Conclusion

In this paper we have given the main definitions and fundamental results needed for looking at specification problems in computer science from the point of view of sketch theory and its generalizations. Our specification tools are the A-wefts for some category A, which is generally projectively sketchable.

Wefts of ambigraphs, i.e.  $\mathcal{A}mbi$ -wefts (where  $\mathcal{A}mbi$  is the category of ambigraphs) are well-suited for dealing with purely functional problems.

However, for dealing with the implicit features of computer languages, we will see in [guide2] and [guide3] that we need a new specification tool. This is given by mosaics, which construction requires  $\mathcal{A}$ -wefts for other categories  $\mathcal{A}$ .

### A Appendix: About categories

Here is a short survey about *compositive graphs*, *categories*, *functors* and *natural transfor-mations*, as well as *projective limits* and *inductive limits* in a category. These notions are well known. However the way limits are introduced here, from *typical cones*, comes from the theory of sketches and wefts. Various examples of all these notions can be found in the previous sections of this paper.

#### A.1 Directed graphs

**Definition 15** A directed graph  $\mathcal{G}$  is made up of:

- a set of points  $\mathcal{G}(Pt)$ ,
- a set of arrows  $\mathcal{G}(Ar)$ ,
- and two maps  $\mathcal{G}(dom)$  and  $\mathcal{G}(codom)$  from  $\mathcal{G}(Ar)$  to  $\mathcal{G}(Pt)$ , which assign to each arrow respectively its *domain* and its *codomain*.

Let g be an arrow of domain  $G_1$  and codomain  $G_2$ ; it is denoted  $g: G_1 \to G_2$  and we say that  $G_1 \to G_2$  is the rank of g. A pair of arrows  $(g_1, g_2)$  is consecutive if the codomain of  $g_1$  is equal to the domain of  $g_2$ . The set of pairs of consecutive arrows is denoted  $\mathcal{G}(\texttt{ConsP})$  and the set of pairs of arrows with the same rank is denoted  $\mathcal{G}(\texttt{RankP})$ .

The points of a directed graph are often called *vertices*, and its arrows *edges*.

Proposition 1 in 3.4 justifies the use of notations like  $\mathcal{G}(Pt)$  etc., rather than  $Pt(\mathcal{G})$  etc. for example.

Often, " $G \in \mathcal{G}$ " means " $G \in \mathcal{G}(Pt)$ ". Also, dom(g) and codom(g) denote the domain and the codomain of an arrow g of  $\mathcal{G}$ , rather than  $\mathcal{G}(dom)(g)$  and  $\mathcal{G}(codom)(g)$  respectively.

A homomorphism of directed graphs preserves the direction of arrows, whereas a  $contravariant\ homomorphism$  reverses it.

**Definition 16** Let  $\mathcal{G}$  and  $\mathcal{G}'$  be two directed graphs.

A homomorphism of directed graphs  $\varphi: \mathcal{G} \to \mathcal{G}'$  is made up of two maps:

$$\left\{ \begin{array}{l} \varphi(\mathtt{Pt}): \mathcal{G}(\mathtt{Pt}) \to \mathcal{G}'(\mathtt{Pt}) \\ \varphi(\mathtt{Ar}): \mathcal{G}(\mathtt{Ar}) \to \mathcal{G}'(\mathtt{Ar}) \end{array} \right.$$

such that for all arrow  $g: G_1 \to G_2$  of  $\mathcal{G}$ :

$$\varphi(\operatorname{Ar})(g): \varphi(\operatorname{Pt})(G_1) \to \varphi(\operatorname{Pt})(G_2) \text{ in } \mathcal{G}'.$$

A contravariant homomorphism of directed graphs  $\varphi: \mathcal{G} \to \mathcal{G}'$  is made up of two maps:

$$\left\{egin{array}{l} arphi(\mathtt{Pt}): \mathcal{G}(\mathtt{Pt}) 
ightarrow \mathcal{G}'(\mathtt{Pt}) \ arphi(\mathtt{Ar}): \mathcal{G}(\mathtt{Ar}) 
ightarrow \mathcal{G}'(\mathtt{Ar}) \end{array}
ight.$$

such that for all arrow  $g: G_1 \to G_2$  of  $\mathcal{G}$ :

$$\varphi(\operatorname{Ar})(g): \varphi(\operatorname{Pt})(G_2) \to \varphi(\operatorname{Pt})(G_1) \text{ in } \mathcal{G}'.$$

The maps  $\varphi(Pt)$  and  $\varphi(Ar)$  are often both denoted  $\varphi$ .

#### A.2 Compositive graphs

**Definition 17** A compositive graph  $\mathcal{G}$  is a directed graph with:

- a set of identity arrows  $\mathcal{G}(\operatorname{IdAr}) \subseteq \mathcal{G}(\operatorname{Ar})$ , such that  $\mathcal{G}(\operatorname{dom})(g) = \mathcal{G}(\operatorname{codom})(g)$ ; when there is only one identity arrow of rank  $G \to G$  it is often denoted  $id_G$ ;
- a set of pairs of composable arrows  $\mathcal{G}(\texttt{CompP}) \subseteq \mathcal{G}(\texttt{ConsP})$ , and a map  $\mathcal{G}(\texttt{comp})$  from  $\mathcal{G}(\texttt{CompP})$  towards  $\mathcal{G}(\texttt{Ar})$  which maps each pair of composable arrows  $(g_1: G_1 \to G_2, g_2: G_2 \to G_3)$  to its composed arrow  $\mathcal{G}(\texttt{comp})(g_1, g_2): G_1 \to G_3$ , which is denoted  $g_2 \circ g_1$ .

**Definition 18** Let  $\mathcal{G}$  and  $\mathcal{G}'$  be two compositive graphs.

A functor  $\varphi: \mathcal{G} \to \mathcal{G}'$  is a homomorphism of directed graphs such that:

- for all identity arrow g of  $\mathcal{G}$ , the arrow  $\varphi(g)$  is an identity arrow of  $\mathcal{G}'$ ;
- for all composable pair  $(g_1, g_2)$  of  $\mathcal{G}$ , the pair  $(\varphi(g_1), \varphi(g_2))$  is a composable pair of  $\mathcal{G}'$  and  $\varphi(g_2 \circ g_1) = \varphi(g_2) \circ \varphi(g_1)$ .

A contravariant functor  $\varphi: \mathcal{G} \to \mathcal{G}'$  is a contravariant homomorphism of directed graphs such that:

- for all identity arrow g of  $\mathcal{G}$ , the arrow  $\varphi(g)$  is an identity arrow of  $\mathcal{G}'$ ;
- for all composable pair  $(g_1, g_2)$  of  $\mathcal{G}$ , the pair  $(\varphi(g_2), \varphi(g_1))$  is a composable pair of  $\mathcal{G}'$  and  $\varphi(g_2 \circ g_1) = \varphi(g_1) \circ \varphi(g_2)$ .

A functor of compositive graphs is called an *extension* if it is an inclusion on the sets of points and on the sets of arrows. Then it is also an inclusion on the set of identity arrows and on the set of composable pairs. This notion of extension is borrowed from the theory of algebraic specifications, it should not be mistaken for the notion of extension in category theory.

#### A.3 Categories

**Definition 19** A category A is a compositive graph where:

- the map  $a \mapsto \mathcal{A}(\mathtt{dom})(a)$  defines a bijection of  $\mathcal{A}(\mathtt{IdAr})$  on  $\mathcal{A}(\mathtt{Pt})$ ; the inverse map  $A \mapsto id_A$  from  $\mathcal{A}(\mathtt{Pt})$  to  $\mathcal{A}(\mathtt{IdAr})$ , is denoted  $\mathcal{A}(\mathtt{selid})$  and called the selection of identities;
- the sets  $\mathcal{A}(CompP)$  and  $\mathcal{A}(ConsP)$  are equal;

such that:

- unitarity:  $a \circ id_{A_1} = a$  and  $id_{A_2} \circ a = a$  for all arrow  $a : A_1 \to A_2$ ;
- associativity:  $(a_3 \circ a_2) \circ a_1 = a_3 \circ (a_2 \circ a_1)$  for all triple of arrows  $(a_1, a_2, a_3)$  such that the pairs  $(a_1, a_2)$  and  $(a_2, a_3)$  are consecutive.

The points of a category are also called *objects*, and its arrows are also called *morphisms* or *homomorphisms*.

An isomorphism from  $A_1$  to  $A_2$  in  $\mathcal{A}$  is an arrow  $a_1:A_1\to A_2$  such that there exists an arrow  $a_2:A_2\to A_1$  inverse of  $a_1$ , i.e. such that  $a_2\circ a_1=id_{A_1}$  and  $a_1\circ a_2=id_{A_2}$ . Then  $A_1$  and  $A_2$  are isomorphic and we write:

$$A_1 \cong A_2$$
.

Since a category is a compositive graph, we may consider the *functors* (resp. the *contravariant functors*) from a compositive graph towards a category or from a category towards a category. The functors from a compositive graph towards a category are sometimes called *diagrams*.

Let  $\mathcal{A}$  be a category. For all point A of  $\mathcal{A}$  there is a contravariant functor:

$$\operatorname{Hom}_{\mathcal{A}}(-,A): \mathcal{A} \longrightarrow \mathcal{S}et$$

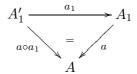
which assigns to each point  $A_1$  of  $\mathcal{A}$  the set:

$$\operatorname{Hom}_{\mathcal{A}}(A_1,A)$$

of arrows from  $A_1$  to A in A, and to each arrow  $a_1: A'_1 \to A_1$  the map:

$$\operatorname{Hom}_{\mathcal{A}}(a_1,A): \operatorname{Hom}_{\mathcal{A}}(A_1,A) \to \operatorname{Hom}_{\mathcal{A}}(A_1',A)$$

defined as  $\operatorname{Hom}_{\mathcal{A}}(a_1,A)(a) = a \circ a_1 : A'_1 \to A$  for all arrow  $a : A_1 \to A$  of  $\mathcal{A}$ .



**Example 22** In fact we have already met three categories (see [ref1] for size issues):

Indeed, the directed graphs with the homomorphisms of directed graphs make up a category  $\mathcal{D}ir$ :

- the identity arrow  $id_{\mathcal{G}}: \mathcal{G} \to \mathcal{G}$  of a directed graph  $\mathcal{G}$  is defined by the two identity maps  $id_{\mathcal{G}(Pt)}$  and  $id_{\mathcal{G}(Ar)}$ ;
- the composed arrow  $\varphi' \circ \varphi$  of a pair of consecutive arrows  $(\varphi, \varphi')$  is defined by the two composed maps  $(\varphi' \circ \varphi)(Pt) = \varphi'(Pt) \circ \varphi(Pt)$  and  $(\varphi' \circ \varphi)(Ar) = \varphi'(Ar) \circ \varphi(Ar)$ .

Similarly:

- the compositive graphs with the functors make up a category *Comp*;
- the categories with the functors make up a category Cat.

#### A.4 Natural transformations

The functors from a compositive graph towards a category can themselves, in a natural way, be seen as the points of a category.

**Definition 20** Let  $\mathcal{G}$  be a compositive graph,  $\mathcal{V}$  a category and  $\mu_1$ ,  $\mu_2$  two functors from  $\mathcal{G}$  to  $\mathcal{V}$ ; a natural transformation  $t: \mu_1 \to \mu_2$  is made up of:

• for all point G of  $\mathcal{G}$ , an arrow  $t(G): \mu_1(G) \to \mu_2(G)$  of  $\mathcal{V}$ ,

such that:

• for all arrow  $g: G_1 \to G_2$  of  $\mathcal{G}$  we have  $t(G_2) \circ \mu_1(g) = \mu_2(g) \circ t(G_1)$  in  $\mathcal{V}$ :

$$\mu_1(G_1) \xrightarrow{t(G_1)} \mu_2(G_1)$$

$$\mu_1(g) \downarrow \qquad = \qquad \downarrow \mu_2(g)$$

$$\mu_1(G_2) \xrightarrow{t(G_2)} \mu_2(G_2)$$

It is also denoted:

$$\mu_1 \left( \stackrel{t}{\Longrightarrow} \right) \mu_2$$

The functors from  $\mathcal{G}$  to  $\mathcal{V}$  are the points and the natural transformations are the arrows of a category:

$$\mathcal{F}unc(\mathcal{G}, \mathcal{V})$$
.

In this category  $\mathcal{F}unc(\mathcal{G}, \mathcal{V})$ :

- the identity arrows are defined from the identity arrows of  $\mathcal{V}$ : for all functor  $\mu: \mathcal{G} \to \mathcal{V}$ , the arrow  $id_{\mu}$  is the natural transformation  $id_{\mu}: \mu \to \mu$  defined by  $id_{\mu}(G) = id_{\mu(G)}$  for all point G of  $\mathcal{G}$ ;
- the composition of arrows is defined from the composition of arrows in  $\mathcal{V}$ : let  $t_1: \mu_1 \to \mu_2$  and  $t_2: \mu_2 \to \mu_3$  be two natural transformations between functors from  $\mathcal{G}$  to  $\mathcal{V}$ , then the composed natural transformation  $t_2 \circ t_1: \mu_1 \to \mu_3$  is defined by  $(t_2 \circ t_1)(G) = t_2(G) \circ t_1(G)$  for all point G of  $\mathcal{G}$ .

From its definition, an isomorphism of the category  $\mathcal{F}unc(\mathcal{G}, \mathcal{V})$  is a natural transformation  $t_1: \mu_1 \to \mu_2$  such that there exists a natural transformation  $t_2: \mu_2 \to \mu_1$  which satisfies  $t_2 \circ t_1 = id_{\mu_1}$  and  $t_1 \circ t_2 = id_{\mu_2}$ , i.e. which satisfies, for all point G of  $\mathcal{G}$ , the equalities  $t_2(G) \circ t_1(G) = id_{\mu_1(G)}$  and  $t_1(G) \circ t_2(G) = id_{\mu_2(G)}$ . In other words,  $t_1$  is an isomorphism of  $\mathcal{F}unc(\mathcal{G}, \mathcal{V})$  if and only if, for all point G of  $\mathcal{G}$ , the arrow  $t_1(G)$  is invertible in  $\mathcal{V}$ . Then  $t_1$  is called a natural isomorphism.

Let  $\mathcal{V}$  be a category. We have defined in A.3 the contravariant functor  $\operatorname{Hom}_{\mathcal{C}omp}(-,\mathcal{V})$ :  $\mathcal{C}omp \longrightarrow \mathcal{S}et$ . From above, for all compositive graph  $\mathcal{G}$ , the set  $\operatorname{Hom}_{\mathcal{C}omp}(\mathcal{G},\mathcal{V})$  is the set of points of the category  $\mathcal{F}unc(\mathcal{G},\mathcal{V})$ . Moreover, it is easy to check that for all functor of compositive graphs  $\varphi: \mathcal{G} \to \mathcal{G}'$ , the map:

$$\operatorname{Hom}_{\operatorname{Comp}}(\varphi, \mathcal{V}) : \operatorname{Hom}_{\operatorname{Comp}}(\mathcal{G}', \mathcal{V}) \to \operatorname{Hom}_{\operatorname{Comp}}(\mathcal{G}, \mathcal{V})$$

is sublying to a functor:

$$\mathcal{F}unc(\varphi, \mathcal{V}): \mathcal{F}unc(\mathcal{G}', \mathcal{V}) \to \mathcal{F}unc(\mathcal{G}, \mathcal{V})$$

which is defined by:

- $\mathcal{F}unc(\varphi, \mathcal{V})(\mu') = \mu' \circ \varphi$  for all functor  $\mu'$  from  $\mathcal{G}'$  to  $\mathcal{V}$ ,
- and  $\mathcal{F}unc(\varphi, \mathcal{V})(t') = t \circ \varphi$  for all natural transformation  $t' : \mu'_1 \to \mu'_2$ , where  $t' \circ \varphi : \mu'_1 \circ \varphi \to \mu'_2 \circ \varphi$  is the natural transformation defined by  $(t' \circ \varphi)(G) = t'(\varphi(G))$  for all point G of  $\mathcal{G}$ .

All this defines a contravariant functor:

$$\mathcal{F}unc(-,\mathcal{V}): \mathcal{C}omp \longrightarrow \mathcal{S}et$$
.

On the other hand, the *equivalence* of two categories, which is weaker than the isomorphism, can be useful.

**Definition 21** An equivalence between two categories  $\mathcal{A}$  and  $\mathcal{A}'$  is a pair of functors  $(\Phi : \mathcal{A} \to \mathcal{A}', \Phi' : \mathcal{A}' \to \mathcal{A})$  and a pair of natural isomorphisms  $\Phi' \circ \Phi \cong id_{\mathcal{A}}$  and  $\Phi \circ \Phi' \cong id_{\mathcal{A}'}$ . It is denoted:

$$A \simeq A'$$
.

**Example 23** A simple example of equivalence which is not an isomorphism is the following one. Let  $\mathcal{A}$  be the category made up of two points  $A_1$  and  $A_2$ , their identities  $id_{A_1}$  and  $id_{A_2}$ , two arrows  $a_1: A_1 \to A_2$  and  $a_2: A_2 \to A_1$ , with the composition defined by  $a_2 \circ a_1 = id_{A_1}$  and  $a_1 \circ a_2 = id_{A_2}$ . Let  $\mathcal{A}'$  be the category made up of one point A' and its identity  $id_{A'}: A' \to A'$ .

Then  $\mathcal{A}$  and  $\mathcal{A}'$  are equivalent: let  $\Phi: \mathcal{A} \to \mathcal{A}'$  be the unique functor from  $\mathcal{A}$  to  $\mathcal{A}'$ , and let  $\Phi': \mathcal{A}' \to \mathcal{A}$  be the functor such that  $A' \mapsto A_1$ . Then of course  $\Phi \circ \Phi' = id_{\mathcal{A}'}$  and  $\Phi' \circ \Phi \neq id_{\mathcal{A}}$ ; it remains to prove that  $\Phi' \circ \Phi \cong id_{\mathcal{A}}$ . For this purpose, let  $t(A_1) = id_{A_1}: A_1 \to A_1$  and  $t(A_2) = a_2: A_2 \to A_1$ . It is easy to check that this defines a natural transformation  $t: id_{\mathcal{A}} \to (\Phi' \circ \Phi)$  and that t is a natural isomorphism. Hence we get the required result:  $\mathcal{A} \cong \mathcal{A}'$ .

In 
$$\mathcal{F}unc(\mathcal{A}, \mathcal{A})$$
:  $id_{\mathcal{A}} \xrightarrow{t} \Phi' \circ \Phi$ 

In  $\mathcal{A}$ :
$$A_{1} \xrightarrow{t(A_{1})=id_{A_{1}}} A_{1}$$

$$a_{1} \xrightarrow{a_{2}} id_{A_{1}} \xrightarrow{id_{A_{1}}} A_{1}$$

$$A_{2} \xrightarrow{t(A_{2})=a_{2}} A_{1}$$

#### A.5 Projective limits

**Definition 22** Let  $\mathcal{B}$  be a compositive graph. The typical projective cone of typical base  $\mathcal{B}$ , also called the typical  $\mathcal{B}$ -projective cone, is the compositive graph  $\mathcal{C}_p(\mathcal{B})$  made up of  $\mathcal{B}$ , a point C, an arrow  $p_B : C \to B$  for all point B of  $\mathcal{B}$ , and an equation  $b \circ p_B = p_{B'}$  for all arrow  $b : B \to B'$  of  $\mathcal{B}$ .

The point C is the vertex of the cone  $C_p(\mathcal{B})$  and the arrow  $p_B$  is its projection towards B.

$$\mathcal{B}: \underbrace{ C_p(\mathcal{B}) : C_{p_B \psi} \underbrace{ C_{p_B \psi$$

**Definition 23** Let  $\mathcal{B}$  and  $\mathcal{G}$  be two compositive graphs. A *projective cone* of typical base  $\mathcal{B}$ , also called a  $\mathcal{B}$ -projective cone, in  $\mathcal{G}$  is a functor  $\chi: \mathcal{C}_p(\mathcal{B}) \to \mathcal{G}$ .

The functor  $\eta = \chi \circ \beta_{\mathcal{B}} : \mathcal{B} \to \mathcal{G}$  is the *base* of the projective cone  $\chi$ . The point  $\chi(C)$  is its *vertex* and the arrows  $\chi(p_B)$  are its *projections*.

$$\mathcal{B} \xrightarrow{\beta_{\mathcal{B}}} \mathcal{C}_p(\mathcal{B})$$

**Definition 24** Let  $\mathcal{B}$  be a compositive graph,  $\mathcal{V}$  a category, and  $L: \mathcal{C}_p(\mathcal{B}) \to \mathcal{V}$  a  $\mathcal{B}$ -projective cone in  $\mathcal{V}$ . Then L is a *limit projective cone* if:

for all projective cone  $P: \mathcal{C}_p(\mathcal{B}) \to \mathcal{V}$  with the same base  $\eta$  as L (i.e. such that  $\eta = P \circ \beta_{\mathcal{B}} = L \circ \beta_{\mathcal{B}}$ ) there is a unique arrow  $\operatorname{projfact}_{L,P}$  (or  $\operatorname{fact}_{L,P}$ ) from P(C) to L(C) in  $\mathcal{V}$  such that  $L(p_B) \circ \operatorname{projfact}_{L,P} = P(p_B)$  for all point B of  $\mathcal{B}$ .

This arrow is the *projective factorisation* (or sometimes simply the *factorisation*) of P with respect to L.

$$B \xrightarrow{b} B'$$

$$\downarrow \qquad \qquad L \downarrow P \downarrow$$

$$L(C) \xrightarrow{projfact_L(P)} P(C)$$

$$\downarrow = \qquad \qquad \downarrow \qquad \qquad$$

Let  $\eta: \mathcal{B} \to \mathcal{V}$ . It is easy to check that two limit projective cones L and L' in  $\mathcal{V}$  with the same base  $\eta$  are isomorphic. Often, one of these limit projective cones is chosen, it is called *the* limit projective cone with base  $\eta$  in  $\mathcal{V}$  and denoted:

$$ProjLim(\eta)$$
 or  $ProjLim(\mathcal{B} \xrightarrow{\eta} \mathcal{V})$ 

or sometimes, omitting the arrows:

$$ProjLim_{B\in\mathcal{B}}(\eta(B))$$
.

The vertex L(C) of the limit projective cone L is often called the *projective limit* of  $\eta$ .

Let us now look at special cases of limit projective cones, which correspond to special typical bases. Notations are the same as above.

**Terminal point.** When the typical base  $\mathcal{B}$  is empty, the projective cone  $\mathcal{C}_p(\mathcal{B})$  is made up of the vertex C alone.

$$\beta_{\mathcal{B}}$$
  $C$ 

Then the vertex  $\mathbb{I} = L(C)$  of L is called a *terminal point* of  $\mathcal{V}$ . Its property can be stated as follows:

• for all point V of V there is a unique arrow:

$$projfact_{\mathbb{L}V}$$
 or  $projfact_V(\ or\ fact_{\mathbb{L}V}\ or\ fact_V): V \to \mathbb{I}$ .

When  $\mathcal{V}$  is the category of sets, it means that  $\mathbb{I}$  is a one-element set.

**Product.** When the typical base  $\mathcal{B}$  is discrete, *i.e.* without any arrow except maybe some identity arrows, the projective cone  $\mathcal{C}_p(\mathcal{B})$  has no non-trivial equation.

Then L is called the *product* of the  $\eta(B)$  (for  $B \in \mathcal{B}$ ) in  $\mathcal{V}$ . Its vertex is denoted  $L(C) = \prod_{B \in \mathcal{B}} \eta(B)$ , or  $L(C) = B_1 \times \ldots \times B_n$  when  $\mathcal{B} = \{B_1, \ldots, B_n\}$ . The property of the product can be stated as follows:

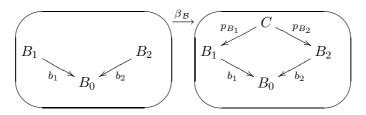
• for all point V of V and all set of arrows  $v_B: V \to \eta(B)$  (for  $B \in \mathcal{B}$ ), there is a unique arrow:

$$v:V \to \prod_{B \in \mathcal{B}} \eta(B)$$

such that  $L(p_B) \circ v = v_B$  for all  $B \in \mathcal{B}$ .

When  $\mathcal{V}$  is the category of sets, it means that  $\prod_{B \in \mathcal{B}} \eta(B)$  is (up to bijection) the cartesian product of the sets  $\eta(B)$ , and the maps  $L(p_B)$  are the canonical projections.

**Pullback.** When the typical base  $\mathcal{B}$  is made up of two arrows with the same codomain, say  $b_1: B_1 \to B_0$  and  $b_2: B_2 \to B_0$  (and maybe some identity arrows), the projective cone  $\mathcal{C}_p(\mathcal{B})$  has three projections  $p_{B_1}: C \to B_1$ ,  $p_{B_2}: C \to B_2$  and  $p_{B_0}: C \to B_0$ , such that  $b_1 \circ p_{B_1} = p_{B_0}$  and  $b_2 \circ p_{B_2} = p_{B_0}$  (so that  $p_{B_0}$  is determined by  $p_{B_1}, p_{B_2}$  and  $\mathcal{B}$ , and can be forgotten).



Then L is called the *pullback* of  $\eta$ , or sometimes the pullback of  $\eta(B_1)$  and  $\eta(B_2)$  above  $\eta(B_0)$ . If we denote  $L(C) = \eta(B_1) \times_{\eta(B_0)} \eta(B_2)$ , the property of the pullback can be stated as follows:

• for all pair of arrows  $v_1: V \to \eta(B_1)$  and  $v_2: V \to \eta(B_2)$  such that  $\eta(b_1) \circ v_1 = \eta(b_2) \circ v_2$ , there is a unique arrow:

$$v: V \to \eta(B_1) \times_{\eta(B_0)} \eta(B_2)$$

such that  $L(p_{B_1}) \circ v = v_1$  and  $L(p_{B_2}) \circ v = v_2$ .

When  $\mathcal{V}$  is the category of sets, it means that  $\eta(B_1) \times_{\eta(B_0)} \eta(B_2)$  is (up to bijection) the set of pairs  $(x_1, x_2) \in \eta(B_1) \times \eta(B_2)$  such that  $v_1(x_1) = v_2(x_2)$  in  $\eta(B_0)$ , and the maps  $L(p_{B_1})$  and  $L(p_{B_2})$  are the canonical projections.

**Monomorphism.** This is a special kind of pullback. With the same typical base  $\mathcal{B}$  as above, let us assume that  $\eta(B_1) = \eta(B_2)$  and  $\eta(b_1) = \eta(b_2)$ . Let  $V_1 = \eta(B_1)$ ,  $V_0 = \eta(B_0)$  and  $m = \eta(b_1) : V_1 \to V_0$ . Let us consider the projective cone L with base  $\eta$  and vertex  $L(C) = V_1$ 

and projections  $L(p_{B_1}) = L(p_{B_2}) = id_{V_1}$  and  $L(p_{B_0}) = m$ .

$$L(B_1) = A_1 \underbrace{L(C) = A_1}_{m} \underbrace{L(B_2) = A_1}_{m}$$

$$L(B_0) = A_0$$

Then it is easy to check that L is the pullback of  $\eta$  if and only if m is a monomorphism, which means:

• for all pair of arrows  $v_1$  and  $v_2: V \to V_1$ :

$$m \circ v_1 = m \circ v_2 \implies v_1 = v_2$$
.

When  $\mathcal{V}$  is the category of sets, it means that m is an injective map.

#### A.6 Inductive limits

Formally, this section is obtained from the previous one by *duality*, *i.e.* by "changing the direction of the arrows" inside the compositive graphs. However, the interpretation of this duality in the category of sets is not straightforward.

**Definition 25** Let  $\mathcal{B}$  be a compositive graph. The typical inductive cone of typical base  $\mathcal{B}$ , also called the typical  $\mathcal{B}$ -inductive cone, is the compositive graph  $\mathcal{C}_i(\mathcal{B})$  made up of  $\mathcal{B}$ , a point C, an arrow  $i_B: B \to C$  for all point B of  $\mathcal{B}$ , and an equation  $i_{B'} \circ b = i_B$  for all arrow  $b: B \to B'$  of  $\mathcal{B}$ .

The point C is the vertex of the cone  $C_i(\mathcal{B})$  and the arrow  $i_B$  is its induction towards B.

$$\mathcal{B}: \overbrace{B \xrightarrow{b} B'} \xrightarrow{\beta_{\mathcal{B}}} C_{i}(\mathcal{B}): \overbrace{C_{i_{\mathcal{B}} \uparrow} = b \atop b} B'$$

**Definition 26** Let  $\mathcal{B}$  and  $\mathcal{G}$  be two compositive graphs. An *inductive cone* of typical base  $\mathcal{B}$ , also called a  $\mathcal{B}$ -inductive cone, in  $\mathcal{G}$  is a functor  $\chi: \mathcal{C}_i(\mathcal{B}) \to \mathcal{G}$ .

The functor  $\eta = \chi \circ \beta_{\mathcal{B}} : \mathcal{B} \to \mathcal{G}$  is the *base* of the inductive cone  $\chi$ . The point  $\chi(C)$  is its *vertex* and the arrows  $\chi(p_B)$  are its *inductions*.

$$\mathcal{B} \xrightarrow{\beta_{\mathcal{B}}} \mathcal{C}_i(\mathcal{B})$$

**Definition 27** Let  $\mathcal{B}$  be a compositive graph,  $\mathcal{V}$  a category, and  $L: \mathcal{C}_i(\mathcal{B}) \to \mathcal{V}$  a  $\mathcal{B}$ -inductive cone in  $\mathcal{V}$ . Then L is a *limit inductive cone* if:

for all inductive cone  $I: \mathcal{C}_i(\mathcal{B}) \to \mathcal{V}$  with the same base  $\eta$  as L, there is a unique arrow  $indfact_{L,P}$  from L(C) to I(C) such that  $indfact_{L,P} \circ L(i_B) \circ I(i_B)$  for all point B of  $\mathcal{B}$ . This arrow is the *inductive factorisation* of I with respect to L.

$$B \xrightarrow{b} B'$$

$$\downarrow \qquad \qquad L \downarrow I \downarrow$$

$$L(C) \xrightarrow{indfact_L(I)} I(C)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad L(B) = I(B) \xrightarrow{L(B')} L(B') = I(B')$$

Let  $\eta: \mathcal{B} \to \mathcal{V}$ . It is easy to check that two limit inductive cones L and L' in  $\mathcal{V}$  with the same base  $\eta$  are isomorphic. Often, one of these limit inductive cones is chosen, it is called *the* limit inductive cone with base  $\eta$  in  $\mathcal{V}$  and denoted:

$$IndLim(\eta)$$
 or  $IndLim(\mathcal{B} \xrightarrow{\eta} \mathcal{V})$ 

or sometimes, omitting the arrows:

$$IndLim_{B \in \mathcal{B}}(\eta(B))$$
.

The vertex L(C) of the limit inductive cone L is often called the *inductive limit* of  $\eta$ . Let us now look at special cases of limit inductive cones.

**Initial point.** When the typical base  $\mathcal{B}$  is empty, the inductive cone  $C_i(\mathcal{B})$  is made up of the vertex C alone.

$$\beta_{\mathcal{B}}$$
  $C$ 

Then the vertex  $\mathbb{O} = L(C)$  of L is called an *initial point* of V. Its property can be stated as follows:

• for all point V of V there is a unique arrow:

$$indfact_{\mathbb{O},V} \ or \ indfact_{V}: \mathbb{O} \to V$$
.

When  $\mathcal{V}$  is the category of sets, it means that  $\mathbb{O}$  is the empty set.

**Sum.** When the typical base  $\mathcal{B}$  is discrete, the inductive cone  $C_i(\mathcal{B})$  has no non-trivial equation.

Then L is called the *sum* of the  $\eta(B)$  (for  $B \in \mathcal{B}$ ) in  $\mathcal{V}$ . Its vertex is denoted  $L(C) = \coprod_{B \in \mathcal{B}} \eta(B)$ , or  $L(C) = B_1 + \ldots + B_n$  when  $\mathcal{B} = \{B_1, \ldots, B_n\}$ . The property of the sum can be stated as follows:

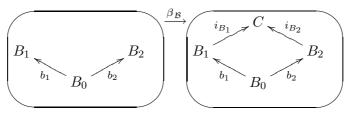
• for all point V of V and all set of arrows  $v_B : \eta(B) \to V$  (for  $B \in \mathcal{B}$ ), there is a unique arrow:

$$v: \coprod_{B\in\mathcal{B}} \eta(B) \to V$$

such that  $v \circ L(i_B) = v_B$  for all  $B \in \mathcal{B}$ .

When  $\mathcal{V}$  is the category of sets, it means that  $\coprod_{B \in \mathcal{B}} \eta(B)$  is (up to bijection) the disjoint union of the sets  $\eta(B)$ , and the maps  $L(i_B)$  are the canonical injections.

**Pushout.** When the typical base  $\mathcal{B}$  is made up of two arrows with the same domain, say  $b_1: B_0 \to B_1$  and  $b_2: B_0 \to B_2$  (and maybe some identity arrows), the inductive cone  $C_i(\mathcal{B})$  has three inductions  $i_{B_1}: B_1 \to C$ ,  $i_{B_2}: B_2 \to C$  and  $i_{B_0}: B_0 \to C$ , such that  $i_{B_1} \circ b_1 = i_{B_0}$  and  $i_{B_2} \circ b_2 = i_{B_0}$  (so that  $i_{B_0}$  is determined by  $i_{B_1}$ ,  $i_{B_2}$  and  $\mathcal{B}$ , and can be forgotten).



Then L is called the *pushout* of  $\eta$ , or sometimes the pushout of  $\eta(B_1)$  and  $\eta(B_2)$  above  $\eta(B_0)$ . If we denote  $L(C) = \eta(B_1) +_{\eta(B_0)} \eta(B_2)$ , the property of the pushout can be stated as follows:

• for all pair of arrows  $v_1: \eta(B_1) \to V$  and  $v_2: \eta(B_2) \to V$  such that  $v_1 \circ \eta(b_1) = v_2 \circ \eta(b_2)$ , there is a unique arrow:

$$v: \eta(B_1) +_{\eta(B_0)} \eta(B_2) \to V$$

such that  $v \circ L(i_{B_1}) = v_1$  and  $v \circ L(i_{B_2}) = v_2$ .

When  $\mathcal{V}$  is the category of sets, it means that  $\eta(B_1) +_{\eta(B_0)} \eta(B_2)$  is obtained (up to bijection) by identifying, in the disjoint union  $\eta(B_1) + \eta(B_2)$ , the elements  $L(i_{B_k})(\eta(b_1)(x))$  and  $L(i_{B_k})(\eta(b_2)(x))$  which come from the same element x of  $\eta(B_0)$ , and the maps  $L(i_{B_1})$  and  $L(i_{B_2})$  are the canonical injections.

**Epimorphism.** This is a special kind of pushout. With the same typical base  $\mathcal{B}$  as above, let us assume that  $\eta(B_1) = \eta(B_2)$  and  $\eta(b_1) = \eta(b_2)$ . Let  $V_1 = \eta(B_1)$ ,  $V_0 = \eta(B_0)$  and  $e = \eta(b_1) : V_0 \to V_1$ . Let us consider the inductive cone L with base  $\eta$  of vertex  $L(C) = V_1$  and inductions  $L(i_{B_1}) = L(i_{B_2}) = id_{V_1}$  and  $L(i_{B_0}) = e$ .

$$L(C) = A_1$$

$$L(B_1) = A_1$$

$$L(B_2) = A_1$$

$$L(B_2) = A_1$$

Then it is easy to check that L is the pushout of  $\eta$  if and only if e is an epimorphism, which means:

• for all pair of arrows  $v_1$  and  $v_2: V_1 \rightarrow V$ :

$$v_1 \circ e = v_2 \circ e \implies v_1 = v_2$$
.

When V is the category of sets, it means that e is a surjective map.

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